Chapter 5

Complex behavior from simple nonlinear feedback[1]

Heikki Hyötyniemi
Helsinki University of Technology, Control Engineering Laboratory

The study of nonlinear dynamic systems is full of buzzwords, and the field is full of wild promises. This chapter discusses these issues – how something surprisingly complex can emerge from seemingly very simple principles, and how easy it is to see something intuitively appealing in the results.

5.1 On Nonlinearity and Feedback

Linear is, intuitively, something that is *straightforward*. From the mathematical point of view, linearity of the function f can be defined in the following way: Function f is linear if

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2) . \tag{5.1}$$

That is, a linear function can be *decomposed*, torn into parts; each part can be manipulated separately, and the final result is the sum of the parts. Summation and scaling does not change the behavior. From the practical point of view, this property is invaluable, facilitating engineering-like, reductionistic approaches.

Another fundamental concept is that of a *dynamical system*. An autonomous, discrete-time dynamic system can be presented using a recursive structure

$$x(k+1) = f(x(k)) , \tag{5.2}$$

[1] The presentation with the same title was given at the STeP Feedback Symposium (Aug. 28, 2000) by Jarmo Hietarinta from the University of Turku

so that the new state $x(k + 1)$ is a linear function of the previous state $x(k)$, where k denotes the time index starting from 0. This means that there is *state feedback* determining the dynamic behavior of the system. This model structure can be visualized as shown in Fig. 5.1.

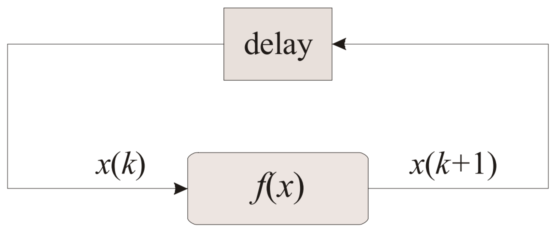


Fig. 5.1. The structure of a dynamic system

More generally, the system state x can be vector-valued; that is, there can be various state variables x_1 to x_n, and the behavior of each of them can be calculated as a function of other state variables:

$$\begin{cases} x_1(k+1) = f_1(x_1(k),\dots,x_n(k)) \\ \vdots \\ x_n(k+1) = f_n(x_1(k),\dots,x_n(k)). \end{cases} \tag{5.3}$$

No matter how high-dimensional the linear model is, that is, no matter what is the value of n, the behavior of the linear system does not become qualitatively more complex. It turns out that the behavior of all the state variables can always be presented as a sum of terms consisting of polynomial, exponential, and harmonic components:

$$x_i(k) = a_1 \cdot k^{b_1} \cdot \mathrm{e}^{c_1 k} \cdot \sin\left(d_1 k + e_1\right) + \cdots + a_m \cdot k^{b_m} \cdot \mathrm{e}^{c_m k} \cdot \sin\left(d_m k + e_m\right), \tag{5.4}$$

where a_1, etc., are fixed parameters, each of them being characteristic to the state variable; m is the number of dynamic *modes*. The range of possible dynamic patterns is also limited to a sum of terms with polynomial, exponential, and harmonic components.

There are very powerful mathematical methodologies developed for analysis and manipulation of linear systems. On the other hand, the strong mathematical machinery collapses altogether if linearity assumption does not hold. In what follows, examples of the complexity of nonlinear dynamics are presented.

5.2 On Bifurcations and Chaos

For a moment, let us study population growth models. The simplest population model consists of just one state variable and linear dynamics:

$$x(k+1) = f(x(k)) = \lambda \cdot x(k). \tag{5.5}$$

The state variable denotes *population density*, and λ is the *growth factor*. This model assumes that the coming population is directly proportional to the current population –

describing exponential, unlimited growth. Indeed, the difference equation can be solved explicitly for x giving

$$x(k) = \lambda^k \cdot x(0) . \tag{5.6}$$

The behavior of the population density is dependent of the model parameter λ. Values below 1 result in exponential decay, finally ending in extinction, whereas values above that result in exponential growth and population explosion (see Fig. 5.2). Even though the population density may behave abruptly, its behavior from now to eternity is exactly predictable.

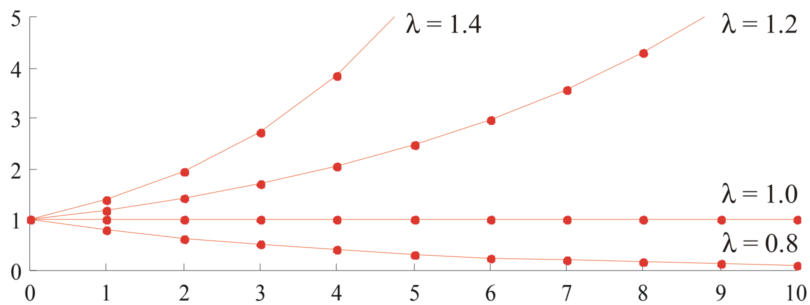


Figure 5.2. Exponential growth/decay

The dynamics of the above model is too simplistic to describe any real population – in practice, no growth can continue indefinitely. That is why, different kinds of nonlinear extensions have been proposed to the basic model; perhaps the best known the following formulation (now it has to be assumed that $0 < x < 1$):

$$\begin{aligned} x(k+1) &= \lambda x(k) \cdot \left(1 - x(k)\right) \\ &= \lambda x(k) - \lambda x^2(k). \end{aligned} \tag{5.7}$$

In addition to the linear part, representing exponential behavior for low population densities, there is a term pushing in the opposite direction: The higher the population is, the stronger the quadratic term resists the growth[2]. Indeed, the population density follows the *logistic curve*, finding some equilibrium population density value after some time has elapsed. However, stable equilibrium is *not necessarily found* – studying the properties of the logistic model results in astonishing observations.

If one simulates the population model starting from different initial values $x(0)$ and for different selections of the parameter λ, it turns out that no matter what is the initial value, the qualitative behavior of the iteration is only dependent of λ. For values $0 < \lambda < 1$, the population tends towards zero, leading to extinction. On the other hand, for

[2] The quadratic nonlinearity can be motivated in various ways. If x is interpreted as the population density, so that x is proportional to the probability of some individual being located in some specific area, then its square x^2 can be interpreted as being proportional to the propability of two individuals meeting each other. And, truly, it has been said that two men meeting means the other getting killed – this was true at least in the case of arcaaic Finns living in the sparsely populated wilderness

values $1 < \lambda < 3$, there is a unique stable fixed point: The population always converges to a value that is a function of λ:

$$x(\infty) = 1 - 1/\lambda. \tag{5.8}$$

However, for values $\lambda > 3$, strange phenomena start taking place. The above fixed point no more remains stable. It is the cycle of period two that starts dominating – that is, no matter what is the initial value of x, the population ends up jumping between two fixed values. This is only the beginning of the story: When the parameter λ grows further, another *bifurcation* takes place, meaning that the period two orbit is split into a stable period four cycle. This period doubling continues until another qualitative surprise takes place: orbits that are not multiples of 2 are detected. It can be shown that before the parameter value $\lambda = 4$ is reached, *all integer periods* have been found; getting towards value $\lambda = 4$, the behavior becomes totally chaotic, with no cycle whatsoever. In Fig. 5.3, the fixed points and orbits are visualized as functions of λ.

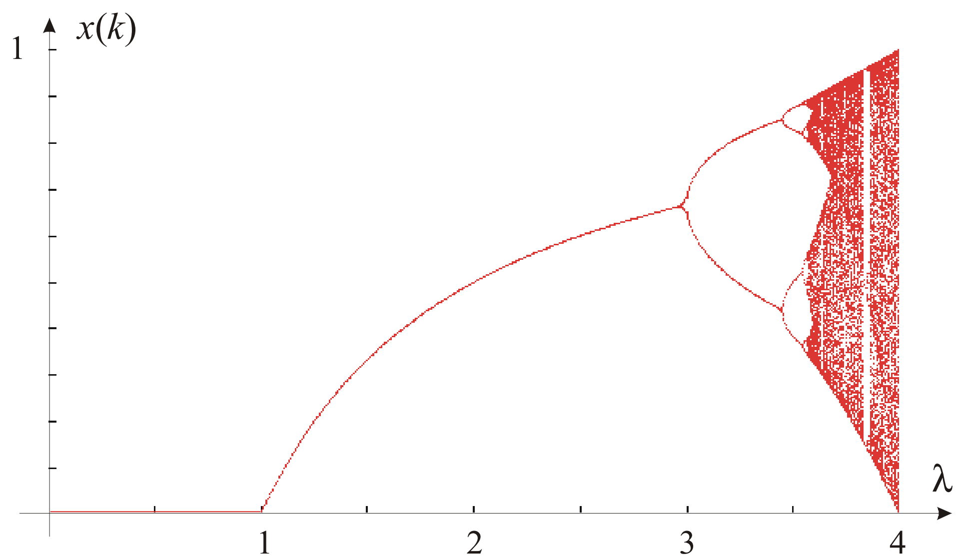
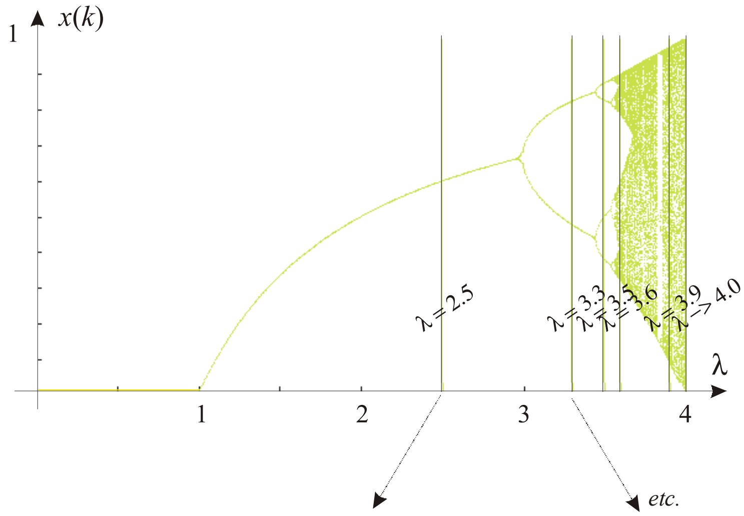


Figure 5.3. "Chaos Icon I": *The Bifurcation Diagram*

In Figure 5.4, the behavior of the iteration is elaborated on further. Some parameter values are selected and resulting typical population dynamics are shown (the iteration has first been run until the stationary behavior has been reached).

What is interesting about this kind of "deterministic chaos" is that even though the parameters and initial conditions may be exactly known, the behavior of the system cannot (in practice) be predicted. This kind of systems are very sensitive to initial conditions: Microscopic perturbations may cause macroscopic effects in the future. This phenomenon has been called the "butterfly effect" – as the weather system is a similarly chaotic system, small actions (the butterfly flying in Amazonas) can cause a hurricane in Japan later that year.



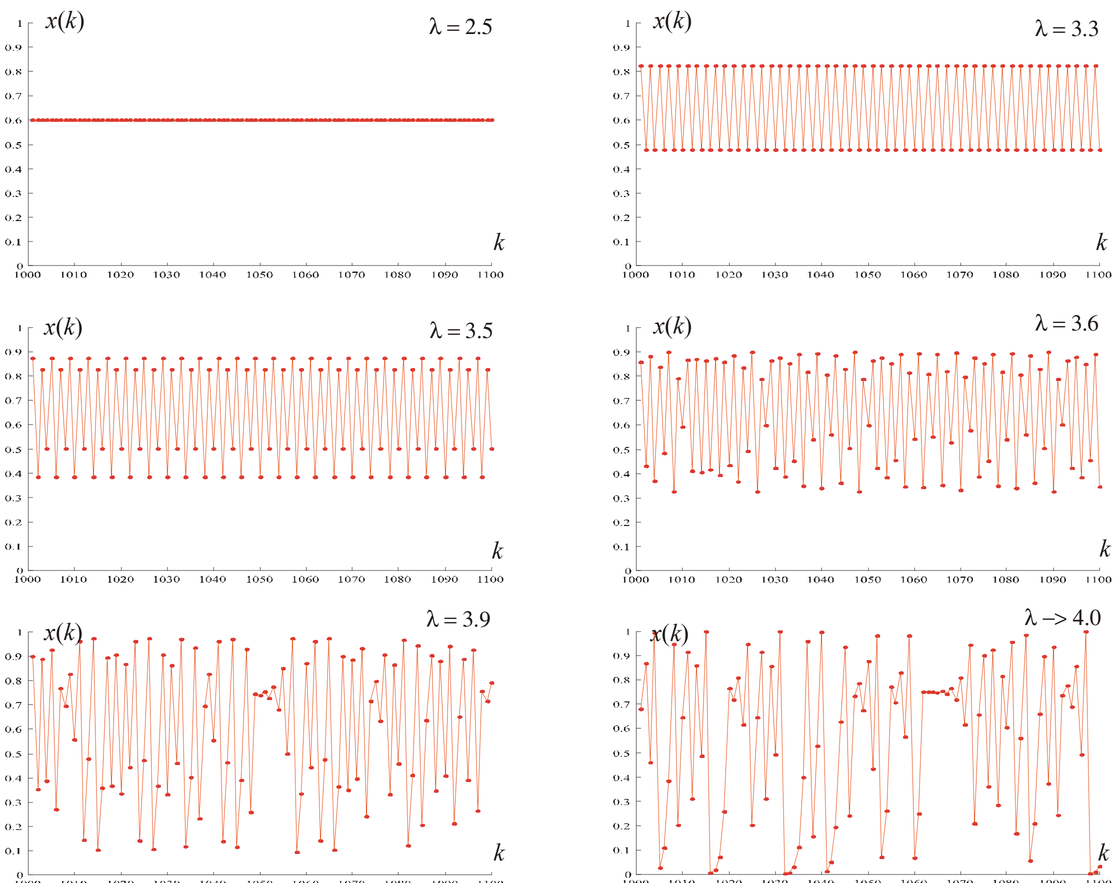


Figure 5.4. Visualization of the bifurcation diagram. The iteration (5.7) has been carried out for six different values of parameter λ. First, there are orbits of period one (fixed point), period two, and period four; the behavior becomes less and less structured until chaos is reached (but note that there still exists some system-specific structure in chaos)

5.3 On Fractals and Self-similarity

Let us study the quadratic iteration from another point of view – study the system model

$$z(k+1) = z^2(k) + z_0 . \tag{5.9}$$

Now it is assumed that the state variable z is a *complex number* ($z = x + y$j, where j is the imaginary unit $\mathrm{j} = \sqrt{-1}$, x and y being real). Exactly the same dynamics as in (5.9) can be presented using pure real values if expressed in the form

$$\begin{cases} x(k+1) = x^2(k) - y^2(k) + x_0 \\ y(k+1) = 2x(k)y(k) + y_0. \end{cases} \tag{5.10}$$

It is really this additional dimension that helps us see new, unanticipated phenomena … however, visualization methods are now different, because the images also have to be drawn in two dimensions. When one simulates (5.10) selecting different values for x_0 and y_0, the iteration starting from $x(0) = y(0) = 0$ may either converge or diverge; when the points resulting in convergence are plotted in black in the (x_0,y_0) plane, one receives the famous *Mandelbrot set* [5.7] – see Fig. 5.5. On the other hand, when some fixed x_0 and y_0 are selected but $x(0)$ and $y(0)$ are varied, the succession of x and y values can be plotted in the (x,y) space – the results are called *Julia sets* (see Fig. 5.6; the bounded orbits are shown in black). These *fractals* (structures of non-integer, fractional dimension) demonstrate *self-similarity*: The same structures repeat in different scales (Fig. 5.7).

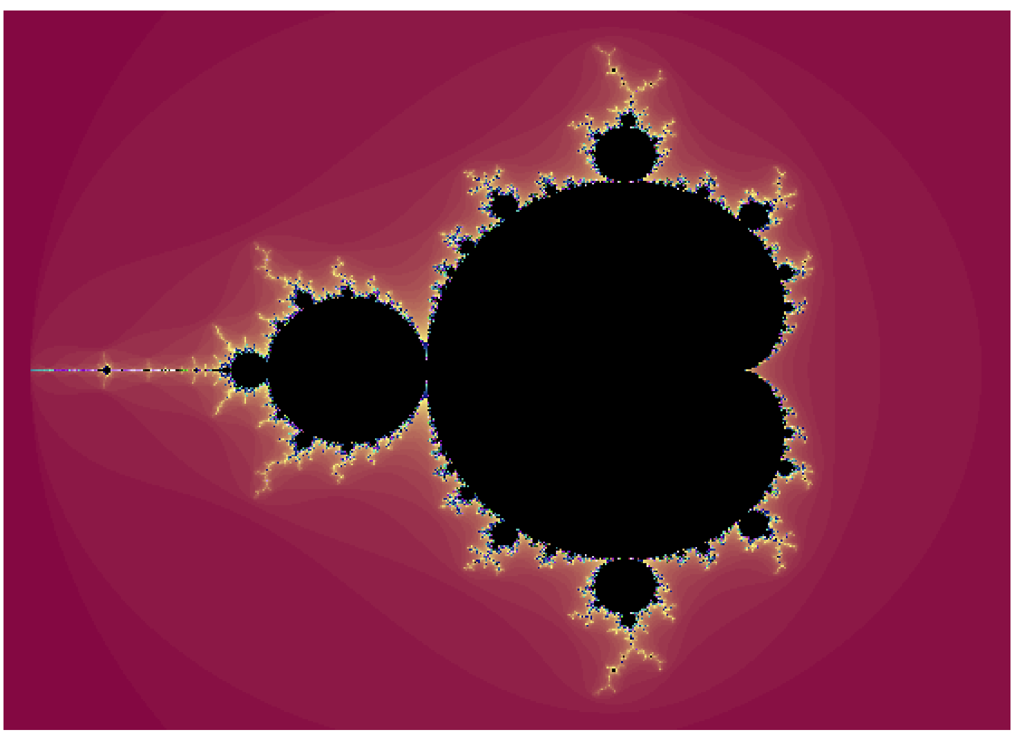


Figure 5.5. "Chaos Icon II": *The Mandelbrot Set*

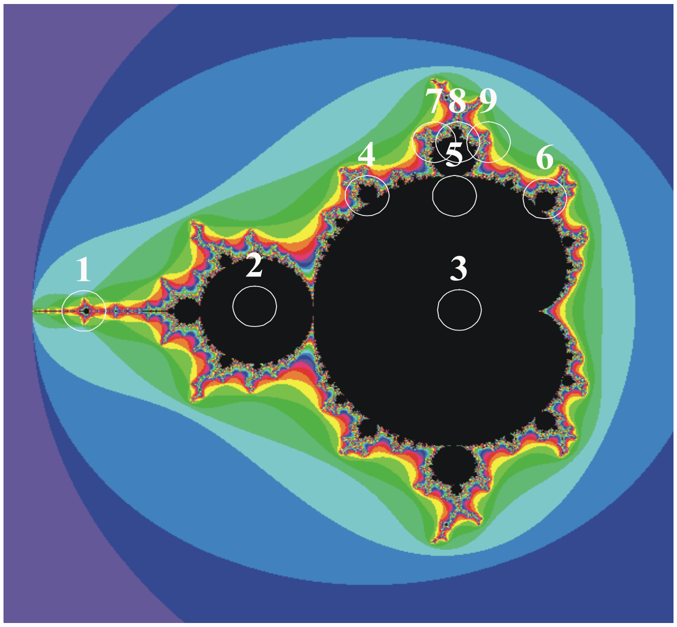
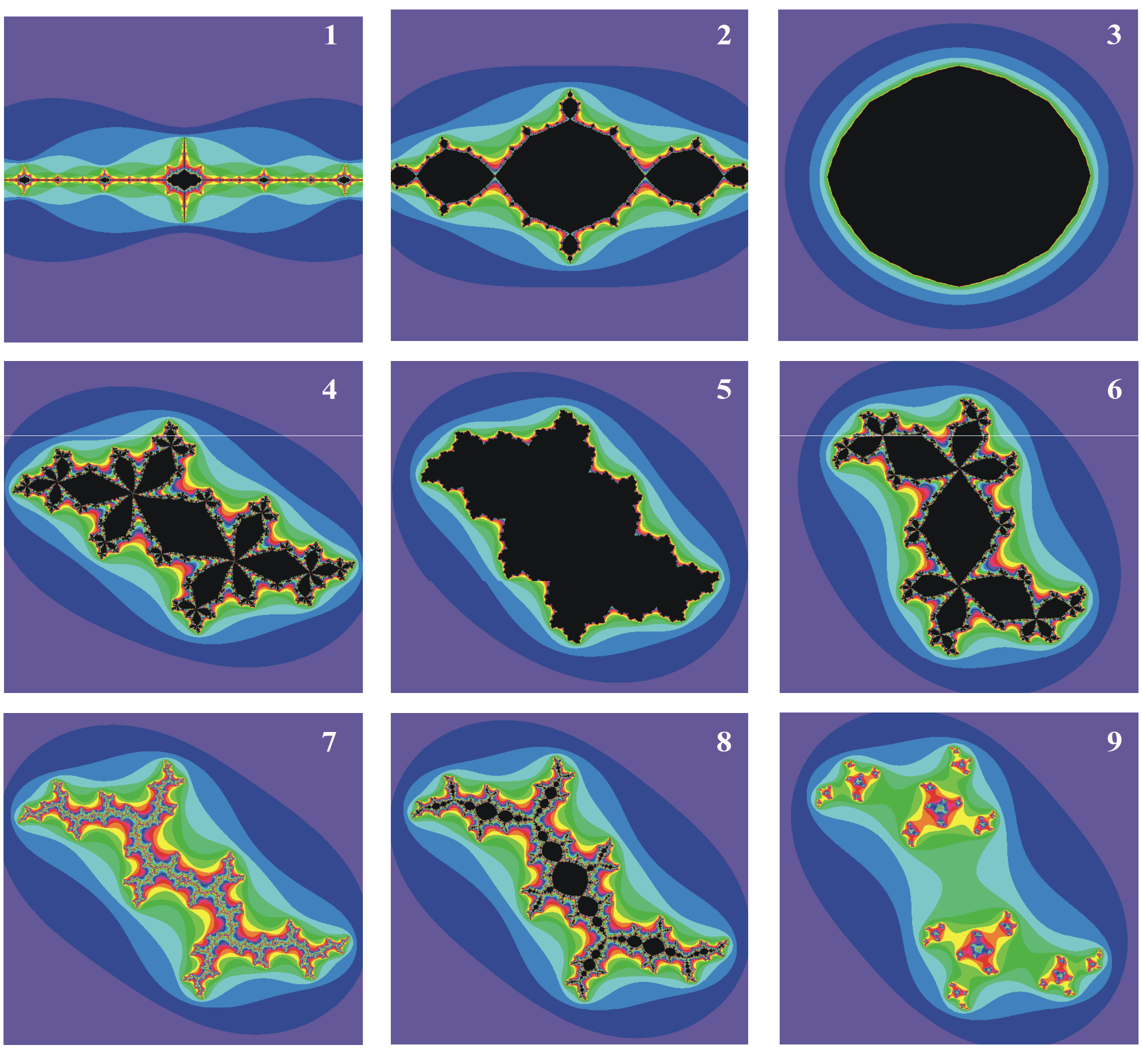
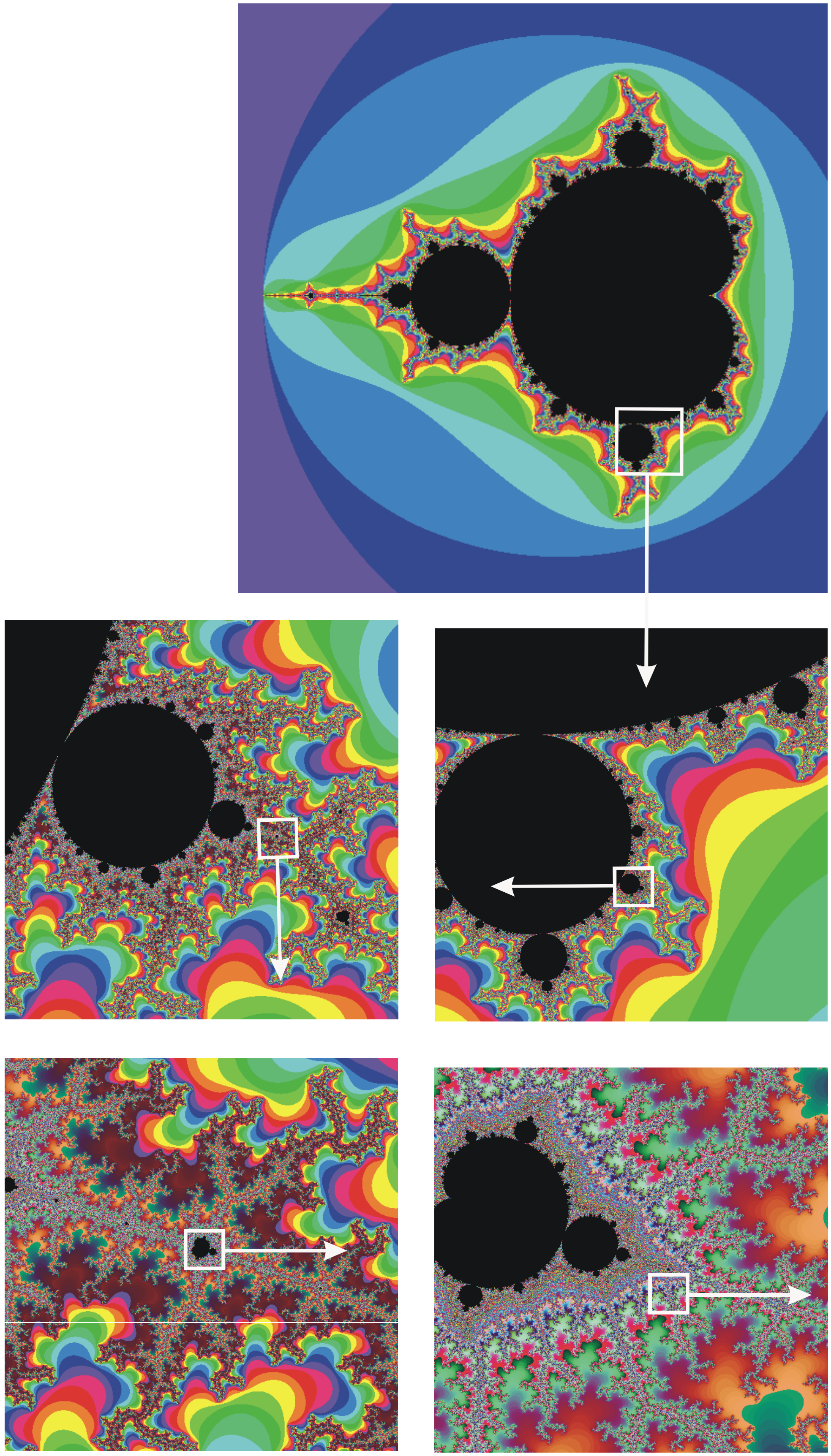


Figure 5.6. Julia sets (below) corresponding to the points in the Mandelbrot set (left)





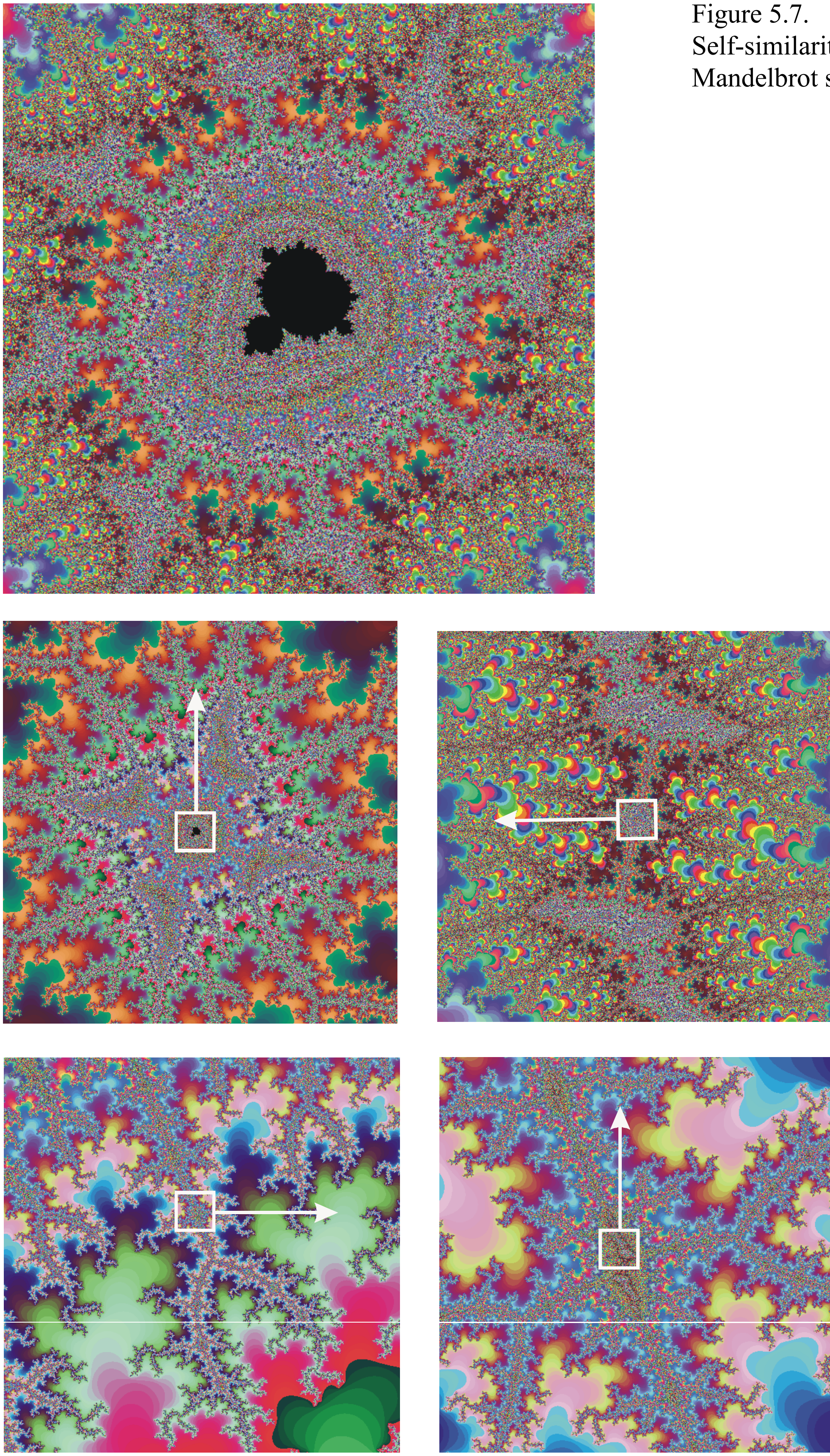


Figure 5.7. Self-similarity of the Mandelbrot set

5.4 On Computability and Universality

After the previous examples, it is clear that nonlinear systems can behave in a most astonishing way. Is there something more to be expected; are there more surprises waiting for us if more complicated nonlinear systems are studied? The answer is *yes* – there are still more qualitative leaps that cannot be predicted when studying the low-dimensional examples. Let us next study a high-dimensional nonlinear model of the form [5.5]

$$\begin{cases} x_1(k+1) = f_{\text{cut}}(a_{11}x_1(k) + \cdots + a_{1n}x_n(k)) \\ \qquad \vdots \\ x_n(k+1) = f_{\text{cut}}(a_{n1}x_1(k) + \cdots + a_{nn}x_n(k)), \end{cases} \tag{5.11}$$

where a_{ij} are constant. The variables are weighted and summed together – apart from f_{cut} the system is linear. This *cut function* is defined so that it eliminates all negative values:

$$f_{\text{cut}}(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

This system structure seems *simpler* than the linear model, the nonlinearity only cutting away the negative values. However, it can be shown that *all computable functions* can be written in the above form. For example, study the generalized XOR problem, or the problem of determining the parity of a given number (whether it is odd or even). It turns out [5.6] that the following realization of (5.11) accomplishes this task (see Fig 5.8):

$$\begin{cases} x_1(k+1) = f_{\text{cut}}(x_1(k) - x_4(k) + x_5(k) - x_7(k) + x_8(k)) \\ x_2(k+1) = f_{\text{cut}}(x_2(k) + x_4(k) - x_5(k) - x_7(k) + x_8(k)) \\ x_3(k+1) = f_{\text{cut}}(x_7(k) - x_8(k)) \\ x_4(k+1) = f_{\text{cut}}(x_3(k)) \\ x_5(k+1) = f_{\text{cut}}(-x_1(k) + x_3(k)) \\ x_6(k+1) = f_{\text{cut}}(x_4(k) - x_5(k)) \\ x_7(k+1) = f_{\text{cut}}(x_6(k)) \\ x_8(k+1) = f_{\text{cut}}(-x_1(k) + x_6(k)). \end{cases} \tag{5.13}$$

When iterating this, starting from

$$\begin{cases} x_1(0) = \text{< given integer to be analyzed >} \\ x_2(0) = 0 \\ x_3(0) = 1 \\ x_4(0) = 0 \\ x_5(0) = 0 \\ x_6(0) = 0 \\ x_7(0) = 0 \\ x_8(0) = 0, \end{cases} \tag{5.14}$$

the system finally converges into a state where the parity of the given integer can be determined, *however large* the initial value of $x_1(0)$ was:

$$\begin{cases} x_1(\infty) = 0 \\ x_2(\infty) = \ < \text{result: 0 for even, 1 for odd input number } x_1(0) > \\ x_3(\infty) = 0 \\ x_4(\infty) = 0 \\ x_5(\infty) = 0 \\ x_6(\infty) = 0 \\ x_7(\infty) = 0 \\ x_8(\infty) = 0. \end{cases} \tag{5.15}$$

The above structure can be interpreted as being a recurrent neural network structure. It is well known that one-layer networks can only perform classification tasks that are linearly separable; using three-layer networks one can, in principle, realize any classifier. However, there is an essential difference here: The more complex the decision surface between the classes becomes, the more one needs neurons; now, on the other hand, the presented network structure with the feedback can be used to implement *qualitatively* more sophisticated tasks. *Infinite number of classes* can be separated from each other using only a *finite network*. Further, because any algorithm can be implemented, feedback structures can also simulate other algorithms; in this sense they can be *universal*. More information on the huge class of computable functions and their strange properties can be found in [5.2]; it turns out that *all formalizable functions* are computable.

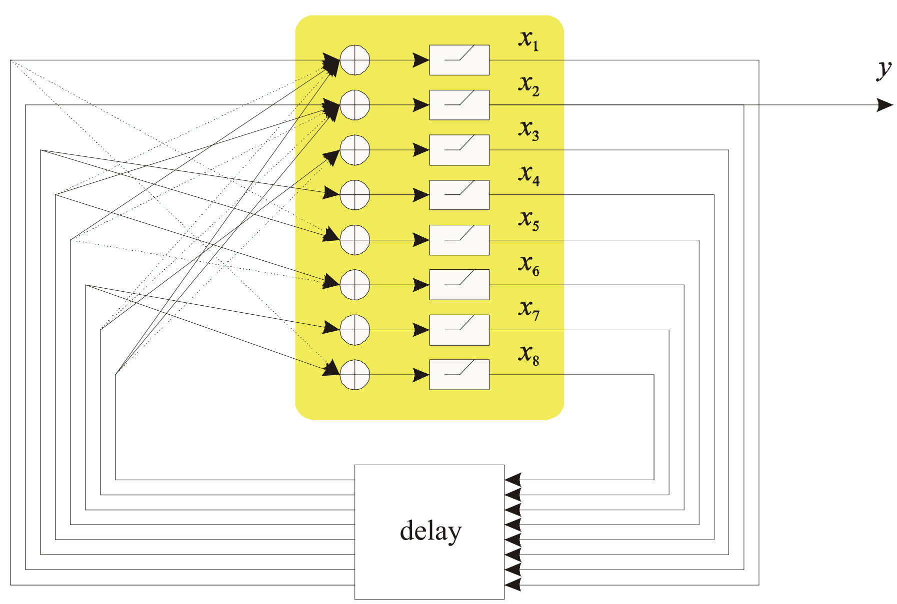


Figure 5.8. The recurrent network implementation of the parity function; solid lines represent “excitatory” and dotted ones “inhibitory” connections

As another example of complex behavior emerging from intuitively simple feedback, study the "Game of Life" as proposed by John Conway [5.1]. The game is based on a *cellular automaton*, operating on an infinite grid, where the following rules apply:

1. A "dead" cell becomes "alive" if it has exactly three living neighbors (out of the total of eight neighbors).
2. A cell remains alive if it has two or three living neighbors.
3. Otherwise, the cell dies (or remains dead).

In Figure 5.9, a typical evolution of cell generations is shown; depending on the initial configuration of living cells, the faith of them is uniquely determined. A more interesting Life form, the "glider", exhausting infinite amount of space but still remaining bounded (!), is shown in Fig. 5.10. What kind of variations of dynamic behavior can emerge in this game? It turns out that there are configurations, for example, that can emit a continuous *train* of gliders. Then some configurations can annihilate gliders, some can change the glider direction, etc. It turns out that, interpreting gliders as binary digits, *logic circuits* operating on gliders can be implemented. Further, *all components of a digital computer* can be constructed. Configuring the initial cells appropriately, it is possible to emulate a computer carrying out execution of some program!

Again, it turns out that in Life *it is possible to design such initial configurations that any computation can be implemented* [5.1]. The universality property seems to be a rule rather than an exception among complex enough nonlinear feedback systems!

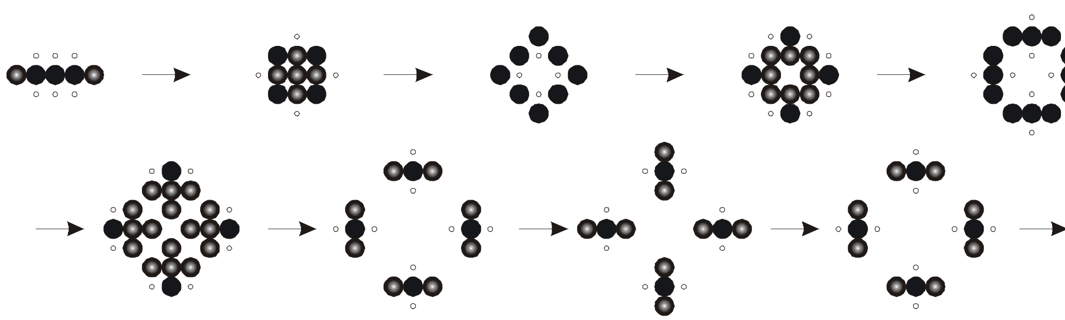


Figure 5.9. Example of Life: The original row of five living cells transforms into a set of four "blinkers", a stable two-period cycle, in six generations. The cells being born are shown as dots and the dying ones are shown in lighter color

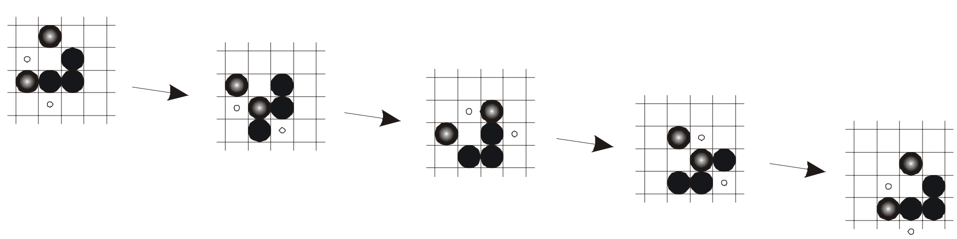


Figure 5.10. The "glider" crawls indefinitely in the diagonal direction, the original shape repeating – slightly shifted – after every four steps

In previous sections it was shown how simple nonlinear feedback systems can *result in unimaginable patterns* – this section revealed, on the other hand, that *any imaginable result can be reached* by simple nonlinear feedback!

5.5 On Complexity and Emergence

The visual nature of the fractal phenomena has been a boost for creative minds for quite a time, and all kinds of wild ideas have been proposed. Chaos and fractality has been seen as the key to understanding *all* complex phenomena. Finding general laws underlying it all – it is like the physicists searching for the "Theory of Everything". The promises are huge:

> "Many of our most troubling long-range problems – trade imbalances, sustainability, AIDS, genetic defects, mental health, computer viruses – center on certain systems of extraordinary complexity. The systems that host these problems – economies, ecologies, immune systems, embryos, nervous systems, computer networks – appear to be as diverse as the problems. Despite appearances, however, the systems do share characteristics, so much so that we group them under a single classification at our Institute, calling them *complex adaptive systems* (*cas*). This is more than terminology. It signals our intuition that there are general principles that govern all cas behavior, principles that point to ways of solving the attendant problems. Much of our work is aimed at turning this intuition into fact."

The above prophecy is given by one of the frontier researchers in "chaoplexity", John Holland [5.3]. There are various theories of how "Complex Adaptive Systems" find their "Emergent Structure" at the "Edge of Chaos" following the ideas of "Self-Organized Criticality".

However, marketing department is here advertising before there is anything to sell: There are no real results supporting the claims yet. As John Horgan puts it [5.4], there are not only big promises but also huge risks. He speaks of "ironic science", meaning scientific work where too many hypotheses are being made and where there are no hard facts supporting them. The potential computational capacity of nonlinear feedback systems (as visualized in Sec. 5.4) makes explicit quantitative analysis of chaotic systems hopeless. The nature of the models is, and will always be descriptive, based on simulations rather that strict theoretical analyses.

Fancy names, buzzwords that do not belong to the mathematical tradition, are used to arouse public interest. What is specially peculiar is that the complex nonlinear feedback processes seem to be working also within the scientific community: The same somewhat sloppy approaches are being generally adopted, following the success of the "hot" disciplines. Horgan even speaks of the "end of science": Astrophysics, for example, one of the most solid branches of scientific work, being based on experiments and hard facts, is turning back to metaphysics, where the "black wormholes" and other subjects that can *never* be quantitatively analyzed are being seriously hypothetized about. Not only chaos research seems to be in chaos.

However, whatever is the final value and role of the new paradigms, nothing will be the same after that. As the Chinese may say to their enemies: “Let you live in interesting times”!

References

[5.1] Berlekamp, E. R., Conway, J. H., and Guy, R. K.: *Winning Ways for Your Mathematical Plays*, Volume 2, chapter 25, Academic Press, New York, 1982, pp. 817–850.

[5.2] Davis, M. and Weyuker, E.: *Computability, Complexity, and Languages – Fundamentals of Theoretical Computer Science.* Academic Press, New York, 1983.

[5.3] Holland, J.: *Hidden Order: How Adaptation Builds Complexity*. Addison–Wesley, Reading, Massachusetts, 1995.

[5.4] Horgan, J.: *The End of Science*. Broadway Books, New York, 1997.

[5.5] Hyötyniemi, H: On unsolvability of nonlinear system stability. In *Proceedings of the European Control Conference (ECC'97)*, Belware Information Technology, Waterloo, Belgium, 1997.

[5.6] Hyötyniemi, H.: Explorations on “Naturalistic Formalisms”. In the *Proceedings of Finnish Artificial Intelligence Days (STeP 2000)*”, Vol. 3, August 2000, pp. 123–131.

[5.7] Mandelbrot, B.: *The Fractal Geometry of Nature*. W. H. Freeman and Company, New York, 1982.

The color version of this text can be found from

- http://saato014.hut.fi/hyotyniemi/publications/01_feedback.htm.

Marvellous Life demonstrations can be found in Internet public domain. For example, the following links were available in the beginning of 2001:

- http://www.rendell.uk.co/gol/tm.htm
- http://psoup.math.wisc.edu/mcell/ca_links.html
- http://www.treasure-troves.com/life/
- http://www.mindspring.com/~alanh/life/

Colorful pages on chaos and Mandelbrot sets, etc., are also available – for example, check the following ones:

- http://aleph0.clarku.edu/~djoyce/julia/explorer.html
- http://archive.comlab.ox.ac.uk/other/museums/computing/mandelbrot.html
- http://www.chaffey.org/fractals/index.html
- http://www.math.utah.edu/~alfeld/math/mandelbrot/mandelbrot.html
- http://calresco.org/sos/sosfaq.htm