Elastic Systems: Another View at Complexity

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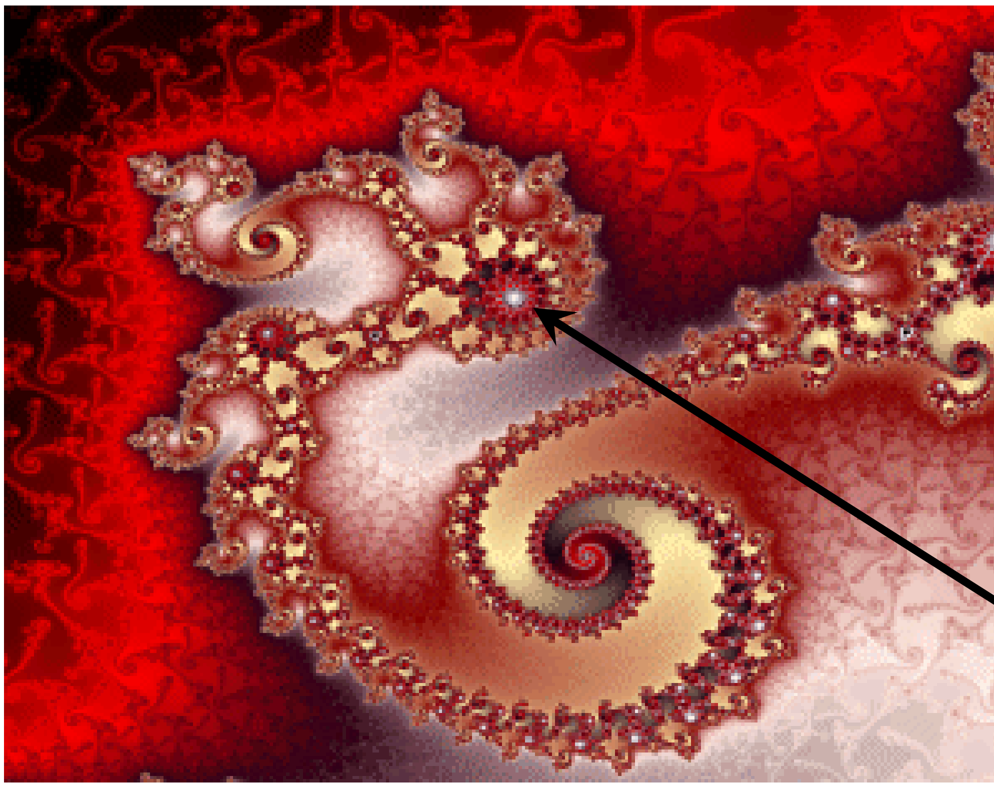
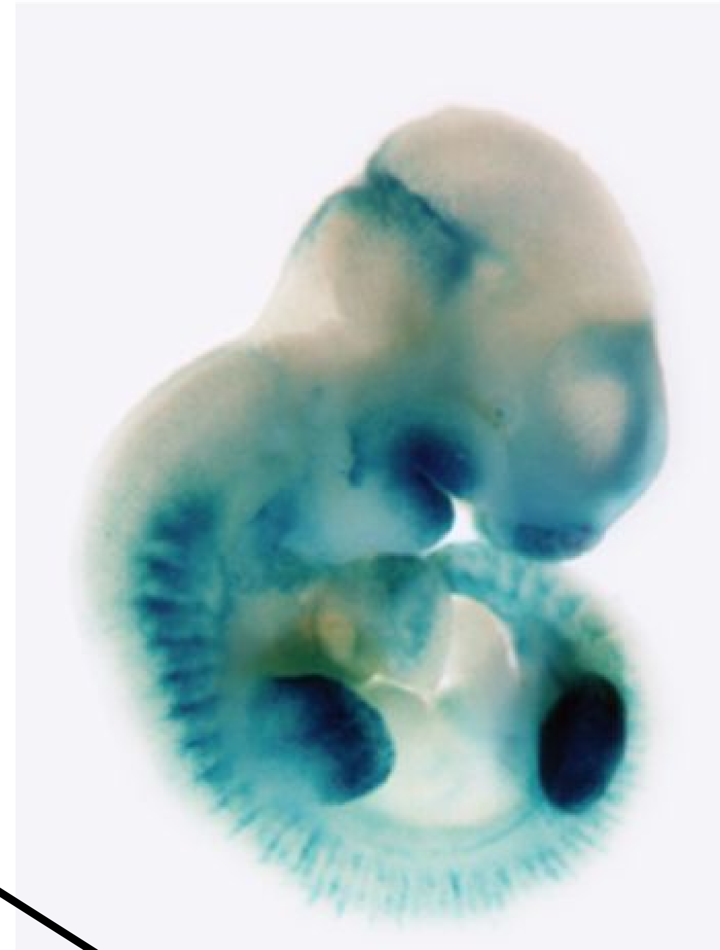
Complexity intuition

- There must be something in common in complex systems!?





Yet, this ”eye” will never see

- ... What is the underlying similarity then?
- Observation: In nature nothing is centrally controlled, everything is completely distributed

Elasticity

= way of looking at (all!) models in distributed perspective

- Nothing very exotic takes place – no ”new science” is needed
- Old science but new interpretations – *new world*
- Apply engineering-like realism and pragmatism = observe and exploit nonidealities, use mathematical tools

Background: ”Ancient Greeks ...”

”You cannot step in the same river twice”

”Everything changes, everything remains the same”

Heraclitus

”Wisdom is knowing how all things are steered by all things”

Panta Rhei!

Everything is based on tensions – and the *hidden tensions* are the *most relevant*

From static pattern to dynamic balance

- Assume the system reacts (linearly) to its environment:

$$0 = \theta^T z$$

Standard way to characterize a system

- Assume that the system is restructured appropriately:

$$A\bar{x} = Bu$$

Tension equilibrium

- Assume that the balance is not yet reached:

$$\frac{dx}{\gamma\, dt} = -Ax + Bu$$

Diffusion process

- For such gradient, there is a *cost* characterizing the system:

$$J = \frac{1}{2} x^T A x - x^T B u$$

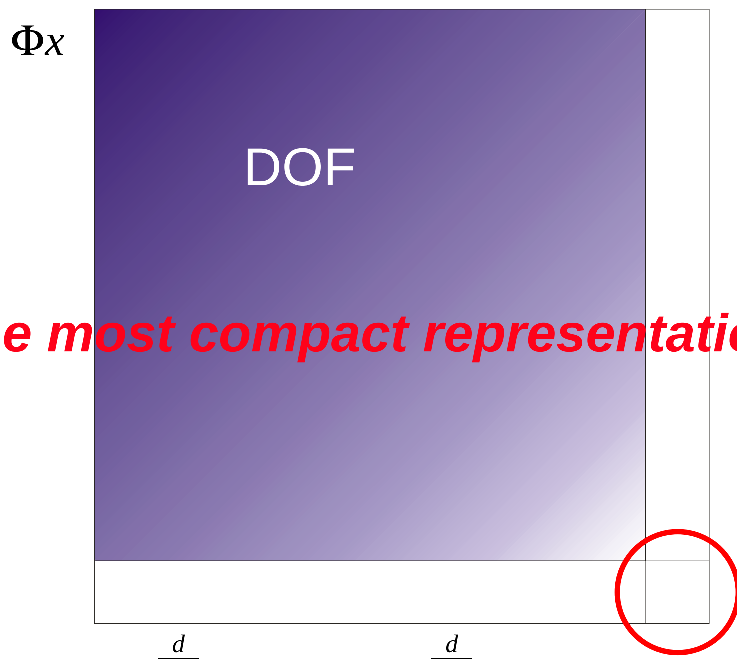
Opposite way to characterize a system!

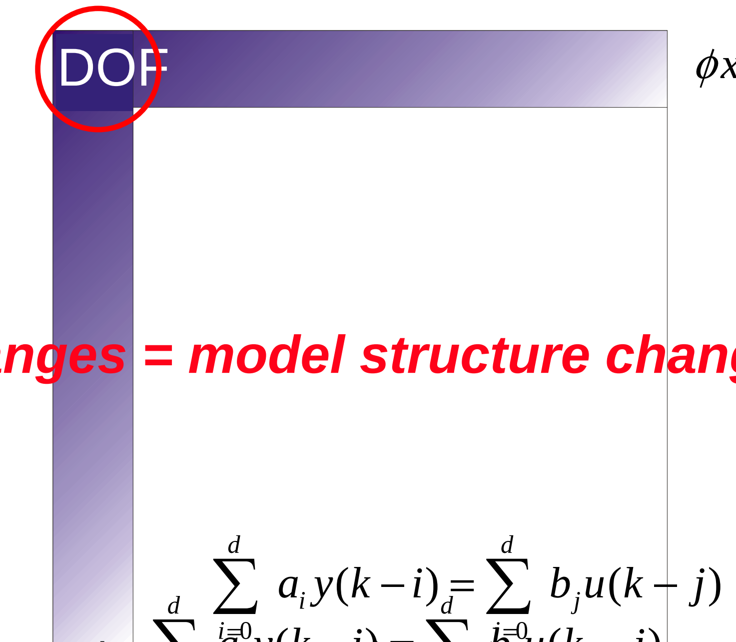
Local linearity can be assumed?!

”Emergent Models”

- Data high-dimensional
- Few constraints
- Many degrees of freedom left

- Data equally high-dimensional
- Many constraints
- Few degrees of freedom (right)





The most compact representation changes = model structure changes

$$\sum_{i=0}^{d} a_i y(k-i) = \sum_{j=0}^{d} b_j u(k-j)$$

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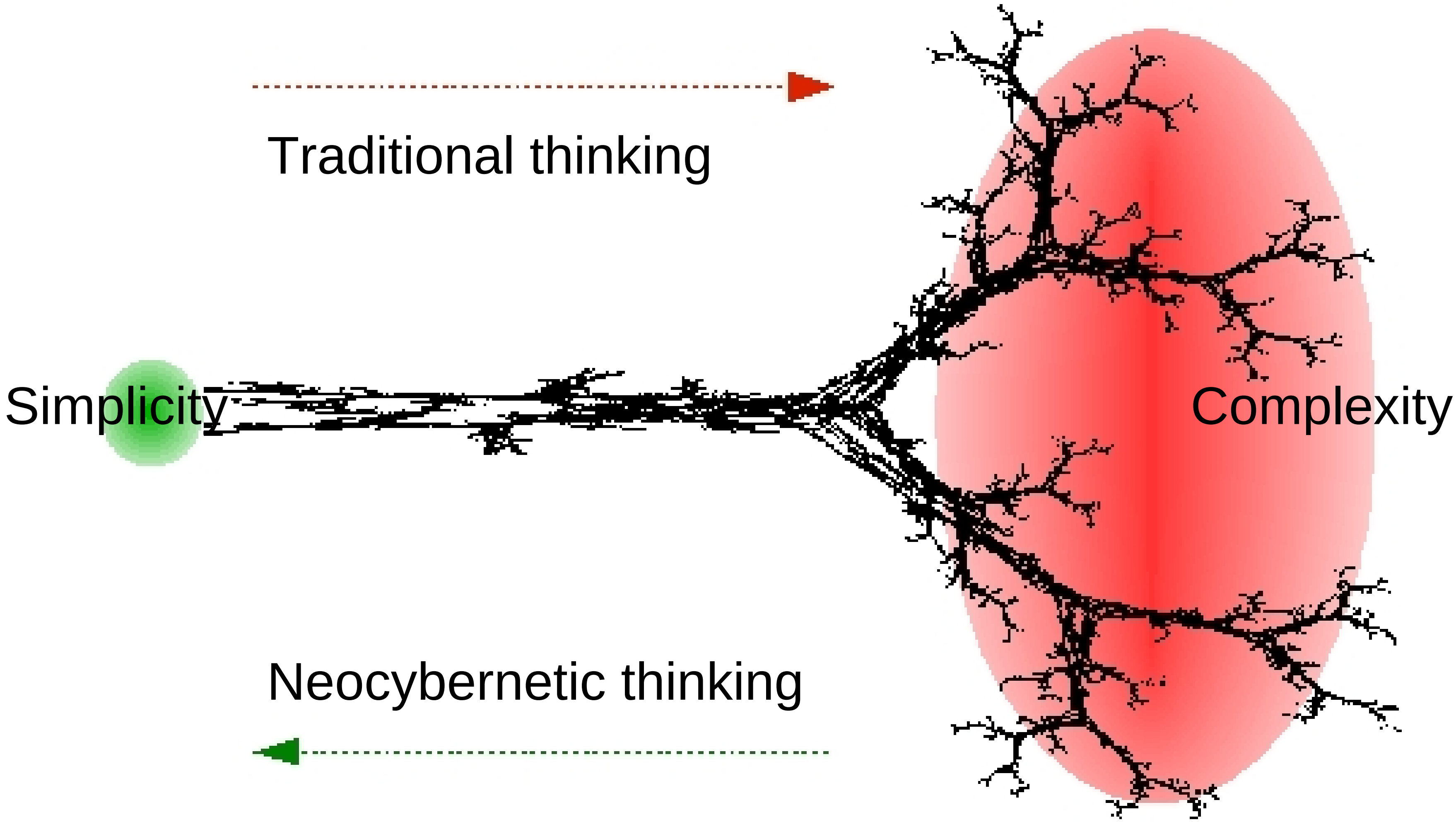
Counterarguments

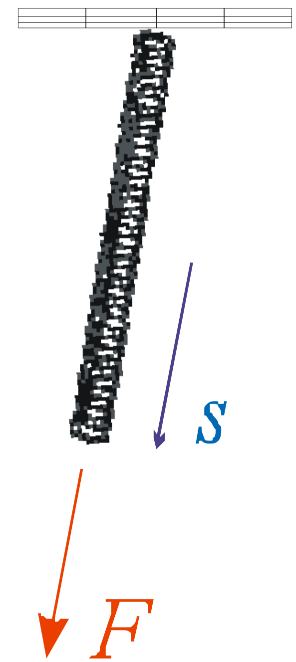
- Complex processes typically consist of chains of actions – how could their behaviors be captured in ”elasticity”?
  - One now only studies stationary, statistical, long-term behaviors where time axis is abstracted away
  - Chains of actions change to coexistence of interactions where all components are connected – one could speak of *pancausality*
- ... But in real systems transient behaviors *are* relevant?
  - One does not try to model all mathematically possible systems now – only physically sustainable systems that can exist – such that do not explode
  - It is assumed that underlying interactions (internal controls) keep the system stable and maintain its integrity – balance essential, details can be ignored
  - ... Not all physically relevant applications are possible in this framework!

Opposite world view



How to interpret the formulas

- Study a one-dimensional case: Spring (spring constant k) stretched (deformation s) by an external force F
- There are *external* and *internal* stored energies in spring (zero level = zero force):



1. Due to the external potential field

$$W_{\text{ext}} = -\int_0^s F\,ds = -Fs$$

2. Due to the internal tensions

$$W_{\text{int}} = \int_0^s ks\,ds = \frac{1}{2}ks^2$$

- Generalization: There are many forces, and many points
- Spring between points s_1 and s_2 (can also be torsional, etc.)

$$W_{\text{int}}(s_1, s_2) = \frac{1}{2} k_{1,2} (s_1 - s_2)^2 = \frac{1}{2} k_{1,2} s_1^2 - k_{1,2} s_1 s_2 + \frac{1}{2} k_{1,2} s_1^2$$

- A matrix formulation is also possible:

$$W_{\text{int}}(s) = \frac{1}{2} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}^T A \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \qquad W_{\text{ext}}(s, F) = - \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}^T B \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}$$

- F_j: Virtual ”generalized forces” as projected along the directions of movements – also torques, shear stresses, etc., all presented in the same framework (for linear structures)

”All” complex systems are elastic systems!

- Now: The difference of potential energies can be expressed as

$$J(s,F) = \frac{1}{2} s^T A s - s^T BF$$

The same cost as found above!

- Here, A is *matrix of elasticity*, and B determines projections
- Matrix A must be symmetric, and must be positive definite to represent stable structures sustaining external stresses
- Principle of minimum potential (deformation) energy: *Structure under pressure ends in minimum of this criterion*
- Elastic systems yield when pressed, but bounce back after it
- Are there additional intuitions available?

Assumption: Goals of local scale actors

- Compare to gravitational field: Potential energy is

  $W_{\text{pot}} = mg\ \Delta h$ *”force times deformation”*

- Elastic system: Average transferred energy / power

  $\mathrm{E}\left\{x_i u_j\right\}$

- Now assume:

  System tries to maximize the coupling with its environment

- That is:

  Maximize the average product of action and reaction

- If this holds for all actors, the system matrices can be written $A = \beta\,\mathrm{E}\{xx^T\}$ and $B = \beta\,\mathrm{E}\{xu^T\}$ for some scalar β

Towards abstraction level #2

- Cybernetic model = statistical model of balances of $x(u)$
- Assume dynamics of u is essentially slower than that of x and study the covariance of $x = \phi^T u = \mathrm{E}\left\{xx^T\right\}^{-1} \mathrm{E}\left\{xu^T\right\} u$

$$\mathrm{E}\left\{xx^T\right\} = \mathrm{E}\left\{xx^T\right\}^{-1} \mathrm{E}\left\{xu^T\right\} \mathrm{E}\left\{uu^T\right\} \mathrm{E}\left\{xu^T\right\}^T \mathrm{E}\left\{xx^T\right\}^{-1}$$

or

$$\mathrm{E}\left\{xx^T\right\}^3 = \mathrm{E}\left\{xu^T\right\} \mathrm{E}\left\{uu^T\right\} \mathrm{E}\left\{xu^T\right\}^T$$

or

$$\left(\phi^T \mathrm{E}\left\{uu^T\right\} \phi\right)^3 = \phi^T \mathrm{E}\left\{uu^T\right\}^3 \phi \qquad n < m$$

- Balance on the statistical level = *second-order balance*

Solution

- Expression fulfilled for $\phi = \theta_n D$, where θ_n is a matrix of n of the covariance matrix eigenvectors, and D is orthogonal

- This is because left-hand side is then

$$\left(\phi^T \mathrm{E}\left\{uu^T\right\}\phi\right)^3 = \left(D^T \theta_n^T \mathrm{E}\left\{uu^T\right\}\theta_n D\right)^3 = \left(D^T \Lambda_n D\right)^3 = D^T \Lambda_n^3 D$$

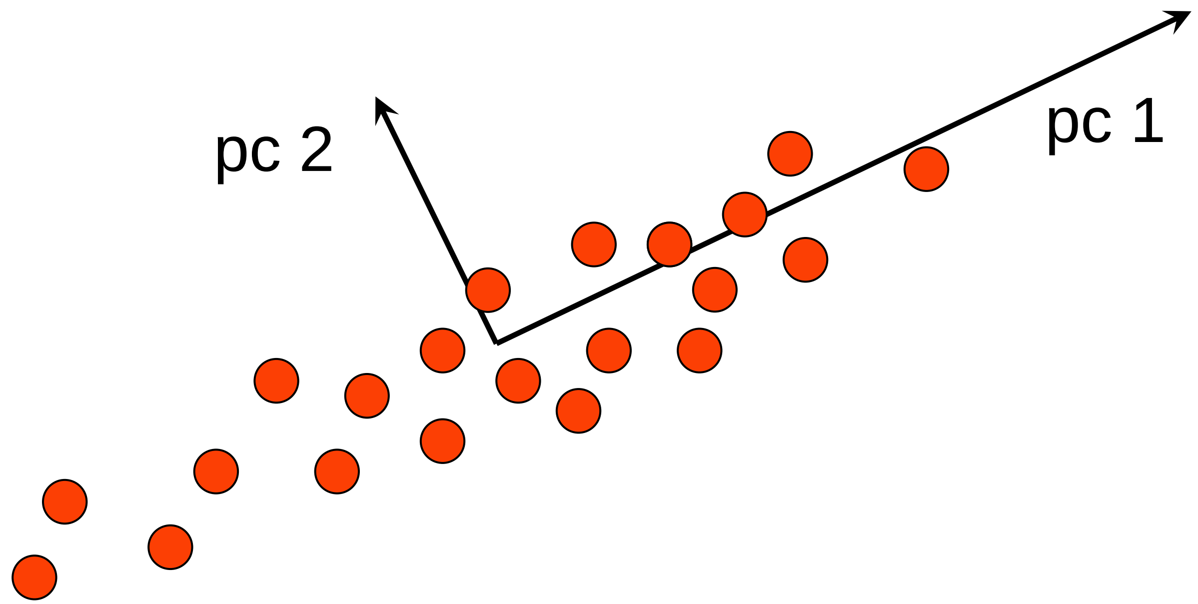
- and right-hand side is

$$\phi^T \mathrm{E}\left\{uu^T\right\}^3 \phi = D^T \theta_n^T \mathrm{E}\left\{uu^T\right\}^3 \theta_n D = D^T \Lambda_n^3 D$$

- Stable solution when θ_n contains the *most significant* data covariance matrix eigenvectors

Principal components

- Principal Component Analysis = Data is projected onto the most significant eigenvectors of the data covariance matrix
- This projection captures maximum of the variation in data
- Principal subspace = PCA basis vectors rotated somehow



Example case: Hebbian learning

- The Hebbian learning rule (by physician Donald O. Hebb) dates back to mid-1900's:

  ”If the neuron activity correlates with the input signal, corresponding synaptic weight increases”

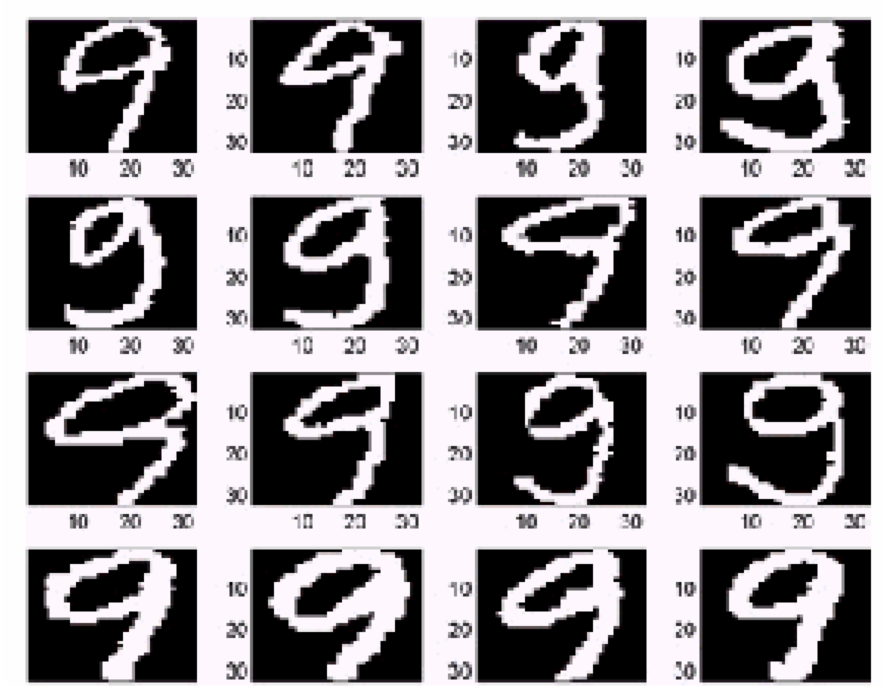
  = Maximize the average product of action and reaction – in the elasticity spirit!

- PCA based modeling of input data takes place in the brain?
- Closer analysis: When stabilization is implemented by feedback through the environment, the principal component axes are rotated towards *sparse components*

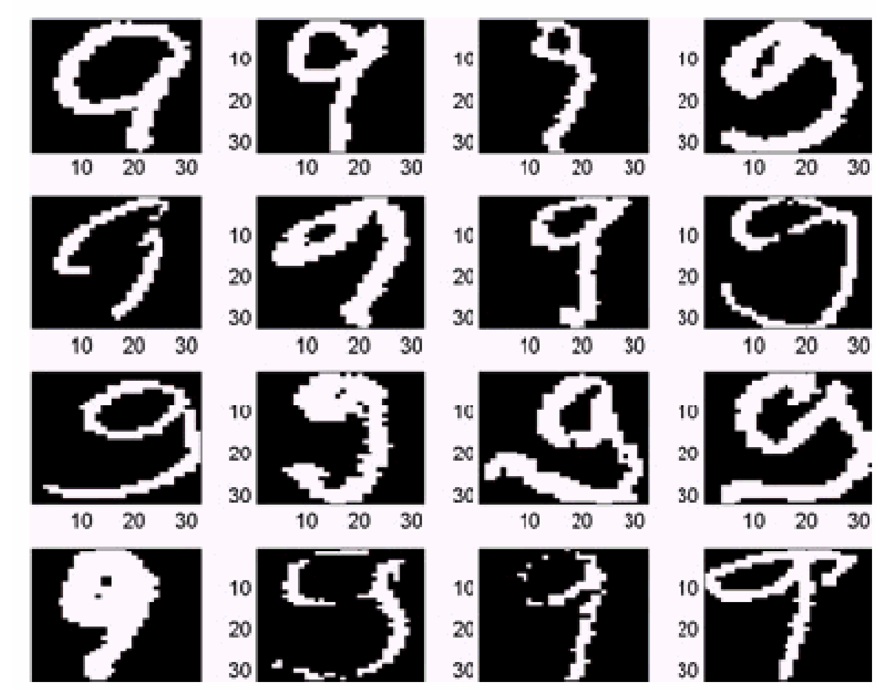
Example: Hand-written digits

- There were a large body of 32x32 pixel images, representing digits from 0 to 9, over 8000 samples (thanks to Jorma Laaksonen)

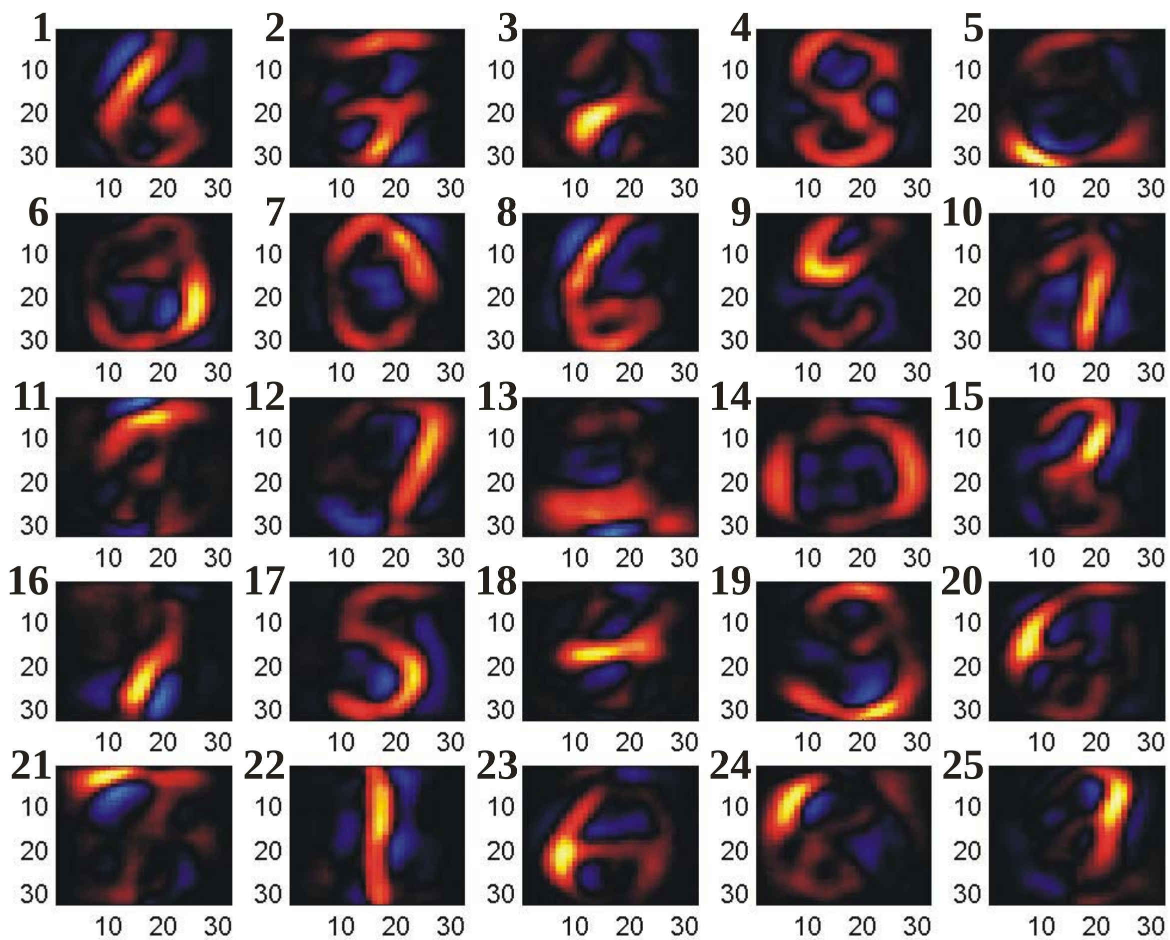
Examples of typical ”9”



Examples of less typical ”9”



... resulting in *sparse components*



Extension to other domains

- Theodosius Dobzhansky: ”Nothing in biology makes sense without reference to evolution”
- Extension: ”Nothing in complex systems *in general* makes sense without reference to evolution”

- It can be claimed that evolutionarily surviving systems implement the derived framework:

  Employing the presented model framework, there is *the best possible exploitation of resources*

- Completely local operation: ”Go towards resources, avoid competition” – but results meaningful on the global scale

Properties of the model

- Robustness.
  - In nature, no catastrophic effects typically take place; even key species are substituted if they become extinct (after a somewhat turbulent period)
  - Now, this can also be explained in terms of the principal subspace: If the profiles are almost orthogonal (PCA-like), disturbances do not cumulate
  - Also because of the principal subspace, sensitivity towards random variations are suppressed
- Biodiversity.
  - In nature, there are many competing species, none of them becoming extinct; modeling this phenomenon seems to be extremely difficult
  - Now, this results from the principal subspace nature of the model: As long as there are various degrees of freedom in input, there are different populations
  - Within populations, this also explains why there exists variation within populations as the lesser principal components also exist.

Pattern matching

- One can also formulate the cost criterion as

$$J(x,u) = \frac{1}{2}(u-\phi x)^T \, \mathrm{E}\left\{uu^T\right\}(u-\phi x)$$

- This means that the neuron grid carries out *pattern matching* of input data
- Note that the traditional maximum (log)likelihood criterion for Gaussian data would be

$$J(x,u) = \frac{1}{2}(u-\phi x)^T \, \mathrm{E}\left\{uu^T\right\}^{-1}(u-\phi x)$$

- **Now:** More emphasis on main directions; no invertibility problems!

PCA is not only *data analysis*
– it can be *system analysis*!

Elastic systems: Summary

- New interpretation of cybernetic systems –
- **”First-order cybernetic system”**
  - Finds balance under external pressures, pressures being compensated by internal tensions
  - Any existing (complex) interacting system that maintains its integrity!
  - Implements minimum *observed* deformation energy along its DOF’s
- **”Second-order cybernetic system”**
  - Adapts the internal structures to better match the observed environmental pressures – towards *maximum experienced stiffness*
  - Any existing (competing) interacting system that has survived in evolution!
  - Implements minimum *average* observed deformation energy

SCAI ”Elastic Systems” session presentations

- **Some theory:**

  - Elastic Systems: Another View at Complexity
  - Elastic Systems: Role of Models and Control (STeP)
  - Elastic Systems: Case of Hebbian Neurons (SCAI poster)

- **Some applications:**

  - Data-Based Modeling of Nickel Plating (STeP)
  - Olli H.: Neocybernetic Modeling of a Biological Cell
  - Kalle H.: Applying Elastic Intuitions to Process Engineering
  - Heikki H.: Emergence in Elastic Sensor / Actuator Networks