Neuron Grids
as seen as
Elastic Systems

Heikki Hyötyniemi
TKK / Control Engineering

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Heikki Hyötyniemi

- Chairman of the Finnish Artificial Intelligence Society (FAIS) 1999 – 2001
- Some 150 scientific publications
- Professor at HUT Control Engineering since Nov. 1, 2001
- Research topic

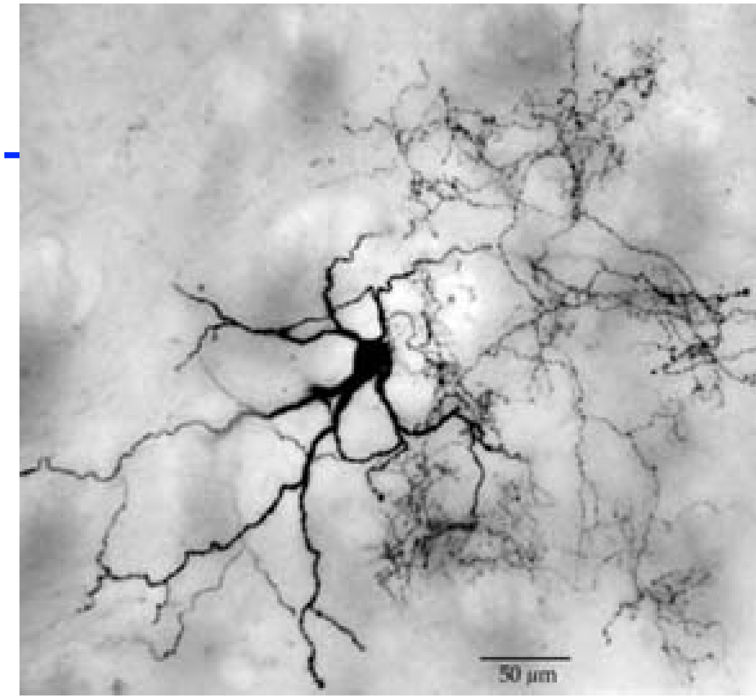
Cybernetic Systems

Neocybernetic starting points – summary

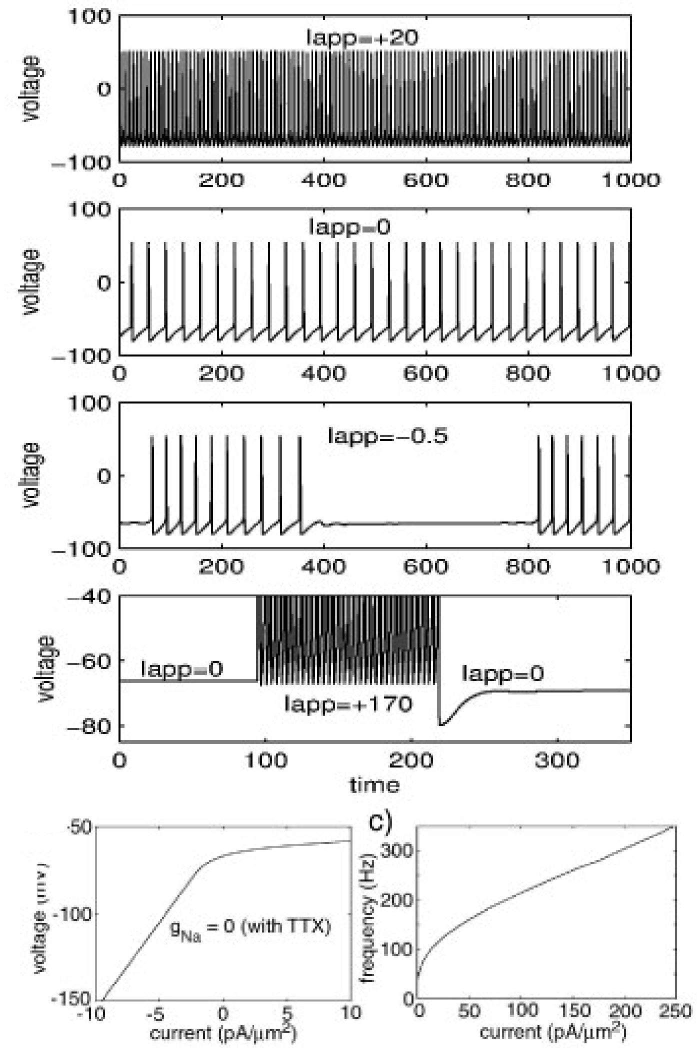
- The details (along time axis) are abstracted away, holistic view from the above is applied
- There exist local actions only, there are no structures of centralized control
- It is assumed that the underlying interactions and feedbacks are consistent, maintaining the system integrity
- This means that one can assume *stationarity* and *dynamic balance* in the system in varying environmental conditions
- An additional assumption: Linearity is pursued as long as it is reasonable

Sounds simple – are there any new intuitions available?

Strong guiding principles for modeling



Modeling a neuron



- Neural (chemical) signals are pulse coded, asynchronous, ... extremely complicated
- Simplification: Only the relevant information is represented – the *activation levels*

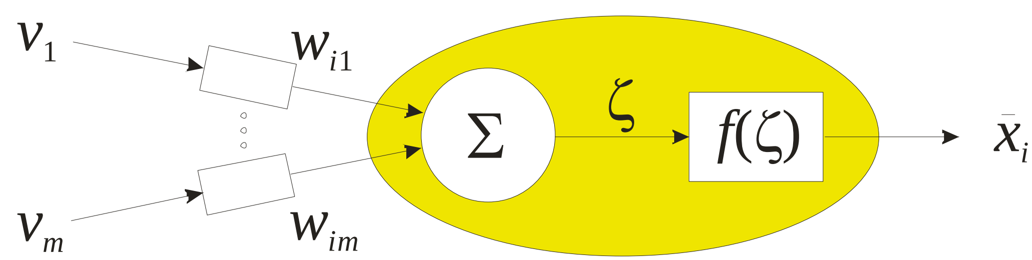
Abstraction level #1

- Triggering of neuronal pulses is stochastic
- Assume that in stationary environmental conditions the average number of pulses in some time interval remains constant
- Only study statistical phenomena: Abstract the time axis away, only model *average activity*
- *Perceptron*: Linear summation of input signals v_j + activation function:

$$\bar{x}_i = f\left(W_i^T v\right)$$

and linear version

$$\bar{x}_i = W_i^T v = \sum_{j=1}^{m} w_{ij}^T v_j$$



For the whole linear grid $\bar{x} = W^T v$

- The emergence idea is exploited here – deterministic activity variables are employed to describe behaviors
- How to exploit the ”first-level” neuron abstraction, how to reach the *neuron grid* level of abstraction?
- Neural networks research studies this – opposite ends:

1. Feedforward perceptron networks
   - Non-intuitive: Black-box model, unanalyzable
   - Mathematically strong: Smooth functions can be approximated to arbitrary accuracy
2. Kohonen’s self-organizing maps (SOM)
   - Intuitive: Easily interpretable by humans (visual pattern recognition capability exploited)
   - Less mathematical: A mapping from m dimensional real-valued vectors to n integers

Yet another approach available?

More general point of view

- Basic mystery: How can the global-level expressions be implemented by the local-level actors?

- Interpret static equations as dynamic equilibria: It is not only noise that can cause deviations from the static model
- Extension gives intuition: Observed constraint is just an emergent pattern – now study the supporting processes

- Basic assumptions:
  - System's responses reflect the environmental pressures
  - Balance of tensions is caused by various counteracting phenomena
  - Balances can be reached through local diffusion processes

From static pattern to a dynamic one

- Assume the system reacts (linearly) to its environment:

$$\bar{x} = \phi^T u$$

- Assume that the system is restructured appropriately:

$$A\bar{x} = Bu$$

- Assume the equality represents a tension equilibrium:

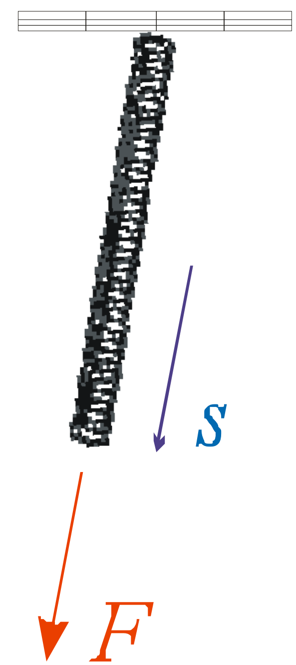
$$\frac{dx}{\gamma\, dt} = -Ax + Bu$$

- For such diffusion, there is a *cost* characterizing the system:

$$J = \frac{1}{2} x^T A x - x^T B u$$

How to interpret

- Study a one-dimensional case: Spring (spring constant k) stretched (deformation s) by an external force F
- There are *external* and *internal* stored energies in spring (zero level = zero force):



1. Due to the external potential field

$$W_{\text{ext}} = -\int_0^s F\,ds = -Fs$$

2. Due to the internal tensions

$$W_{\text{int}} = \int_0^s ks\,ds = \frac{1}{2}ks^2$$

- Generalization: There are many forces, and many points
- Spring between points s_1 and s_2 (can also be torsional, etc.)

$$W_{\text{int}}(s_1, s_2) = \frac{1}{2} k_{1,2} \left(s_1 - s_2\right)^2 = \frac{1}{2} k_{1,2} s_1^2 - k_{1,2} s_1 s_2 + \frac{1}{2} k_{1,2} s_1^2$$

- A matrix formulation is also possible:

$$W_{\text{int}}(s) = \frac{1}{2} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}^T A \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \qquad W_{\text{ext}}(s, F) = -\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}^T B \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}$$

- F_j: Virtual ”generalized forces” as projected along the directions of movements – also torques, shear stresses, etc., all presented in the same framework (for linear structures)

”All” complex *reasonable* systems are elastic!

- Now: The difference of potential energies can be expressed as

$$J(s,F) = \frac{1}{2} s^T A s - s^T B F$$

- Here, A is *matrix of elasticity*, and B determines projections
- Matrix A must be symmetric, and must be positive definite to represent stable structures sustaining external stresses
- Principle of minimum potential (deformation) energy: *Structure under pressure ends in minimum of this criterion*
- Elastic systems yield when pressed, but bounce back after it
- Are there additional intuitions available?

Goals of evolution – local scale

- Compare to gravitational field: Potential energy is

  $W_{\mathrm{pot}} = mg\,\Delta h$ *”force times deformation”*

- Elastic system: Average transferred energy / power

  $\mathrm{E}\left\{\bar{x}_i u_j\right\}$

- Now assume:

  System tries to maximize the coupling with its environment

- That is:

  Maximize the average product of action and reaction

- Special case: *Neuronal system and Hebbian learning seem to implement this principle*

Hebbian learning

- The Hebbian learning rule (by physician Donald O. Hebb) dates back to mid-1900's:

”If the neuron activity correlates with the input signal, the corresponding synaptic weight increases”

- Are there some *goals* for neurons included here? Is there something teleological taking place?
- Bold assumptions make it possible to reach powerful models

Traditional Hebbian learning

- Assume: Perceptron activity $\bar{x}_i$ is a linear function of the input signal v_j, where the vector w_{ij} contains the synaptic weight:

$$\bar{x}_{ij} = w_{ij} v_j \qquad \text{with} \qquad \bar{x}_i = \sum_{j=1}^{m} \bar{x}_{ij}$$

- Hebbian law applied in adaptation: Correlation between input and neuronal activity expressed as $\bar{x}_i\, v_j$, so that

$$\frac{dw_{ij}}{dt} = \gamma \cdot \bar{x}_i v_j = \gamma \cdot w_{ij} v_j^2$$

assuming here, for simplicity, that $m = 1$.

- This learning law is unstable – the synaptic weight grows infinitely, and so does $\bar{x}_i$!

Enhancements

- Stabilization by the *Oja's rule* (by Erkki Oja):

$$\frac{dw_{ij}}{dt} = \gamma \cdot w_{ij} v_j^2 - \gamma \cdot w_{ij} \bar{x}_i^2$$

Compare to the logistic formulation of limited growth!

- Motivation: Keeps the weight vector bounded ($|W_i| = 1$), and average signal size $\mathrm{E}\{|\bar{x}_i|\} = 1$
- Extracts the *first principal component* of the data
- Extension: Generalized Hebbian Algorithm (GHA): Structural tailoring makes it possible to deflate pc's one at a time
- However, the new formula is nonlinear: Analysis of neuron grids containing such elements is difficult, and extending them is equally difficult – What to do instead?

Level of synapses

- The neocybernetic guidelines are: Search for *balance* and *linearity*
- Note: Nonlinearity was not included in the original Hebbian law – it was only introduced for pragmatic reasons

Are there other ways to reach stability – in linear terms?

- Yes – one can apply *negative feedback*:

$$\frac{dw_{ij}}{dt} = \gamma_i \cdot \bar{x}_i v_j - \frac{1}{\tau_i} w_{ij}$$ or in matrix form $$\frac{dW}{dt} = \gamma \cdot \bar{x} v^T - \tau^{-1} W$$

The steady-state is

$$\bar{W} = \gamma\tau \cdot \mathrm{E}\left\{ \bar{x} v^T \right\} = \Gamma \cdot \mathrm{E}\left\{ \bar{x} v^T \right\}$$

Synaptic weights can be coded in a correlation matrix

Level of neuron grids

- Just the same principles can be applied when studying the neuron grid level – *balance* and *linearity*
- Define

$$\bar{W} = \left(A \mid B \right) \quad \text{and} \quad v = \begin{pmatrix} -x \\ \hline u \end{pmatrix}$$

so that $A = \Gamma \cdot \mathrm{E}\left\{ \overline{x x^T} \right\}$ and $B = \Gamma \cdot \mathrm{E}\left\{ \bar{x} u^T \right\}$

- To implement negative feedback, one needs to apply the *anti-Hebbian* action between otherwise Hebbian neurons:

$$\frac{dx}{dt} = -Ax + Bu$$

so that the steady state becomes

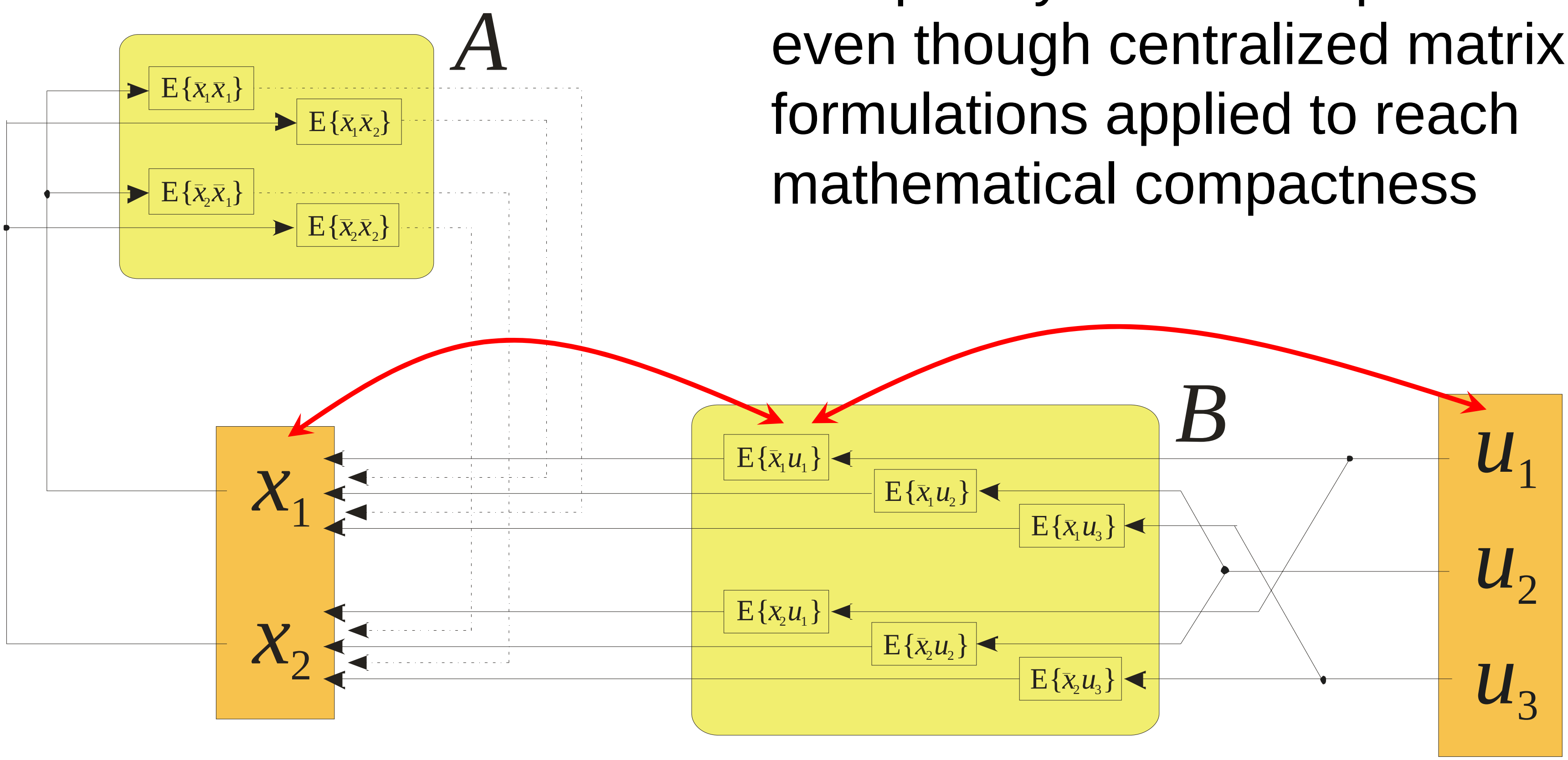
$$\bar{x} = A^{-1} B\, u = \mathrm{E}\left\{ \overline{x x^T} \right\}^{-1} \mathrm{E}\left\{ \bar{x} u^T \right\} u = \phi^T u$$

Model is stable! Eigenvalues of A always real and non-negative

Hebbian/anti-Hebbian system

$$\dot{x} = -Ax + Bu$$

- Explicit feedback structures
- Completely localized operation, even though centralized matrix formulations applied to reach mathematical compactness



Towards abstraction level #2

- Cybernetic model = statistical model of balances $\bar{x}(u)$
- Assume dynamics of u is essentially slower than that of x and study the covariance properties:

$$\mathrm{E}\left\{\bar{x}\bar{x}^T\right\} = \mathrm{E}\left\{\bar{x}\bar{x}^T\right\}^{-1} \mathrm{E}\left\{\bar{x}u^T\right\} \mathrm{E}\left\{uu^T\right\} \mathrm{E}\left\{\bar{x}u^T\right\}^T \mathrm{E}\left\{\bar{x}\bar{x}^T\right\}^{-1}$$

or

$$\mathrm{E}\left\{\bar{x}\bar{x}^T\right\}^3 = \mathrm{E}\left\{\bar{x}u^T\right\} \mathrm{E}\left\{uu^T\right\} \mathrm{E}\left\{\bar{x}u^T\right\}^T$$

or

$$\left(\phi^T \mathrm{E}\left\{uu^T\right\}\phi\right)^3 = \phi^T \mathrm{E}\left\{uu^T\right\}^3 \phi \qquad n < m$$

- Balance on the statistical level = *second-order balance*

Solution

- Expression fulfilled for $\phi = \theta_n D$, where θ_n is a matrix of n of the covariance matrix eigenvectors, and D is orthogonal

- This is because left-hand side is then

$$\left(\phi^T \mathrm{E}\left\{uu^T\right\}\phi\right)^3 = \left(D^T \theta_n^T \mathrm{E}\left\{uu^T\right\}\theta_n D\right)^3 = \left(D^T \Lambda_n D\right)^3 = D^T \Lambda_n^3 D$$

- and right-hand side is

$$\phi^T \mathrm{E}\left\{uu^T\right\}^3 \phi = D^T \theta_n^T \mathrm{E}\left\{uu^T\right\}^3 \theta_n D = D^T \Lambda_n^3 D$$

- Stable solution when θ_n contains the *most significant* data covariance matrix eigenvectors

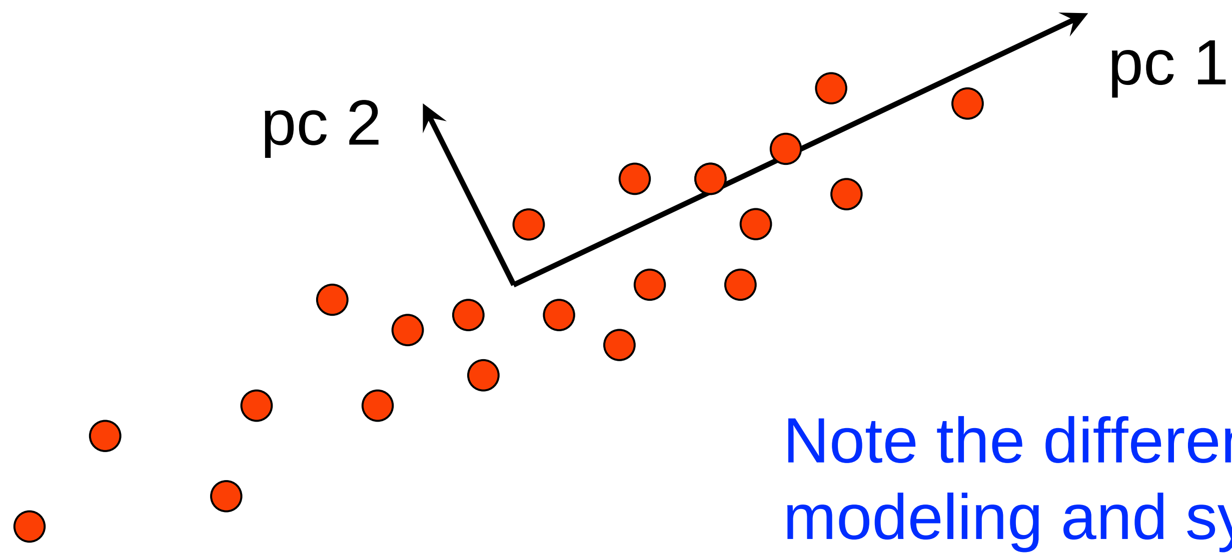
Principal subspace analysis

- Any subset of input data principal components can be selected for ϕ
- The subspace spanned by the n most significant principal components gives a stable solution
- Conclusion:

  Competitive learning (combined Hebbian and anti-Hebbian learning) without any structural constraints results in self-regulation (balance) and self-organization (in terms of principal subspace).

Principal components

- Principal Component Analysis = Data is projected onto the most significant eigenvectors of the data covariance matrix
- This projection captures maximum of the variation in data
- Principal subspace = PCA basis vectors rotated somehow



Note the difference between data modeling and system modeling!

Pattern matching

- One can also formulate the cost criterion as

$$J(x,u) = \frac{1}{2}(u - \phi x)^T \, \mathrm{E}\left\{uu^T\right\}(u - \phi x)$$

- This means that the neuron grid carries out *pattern matching* of input data
- Note that the traditional maximum (log)likelihood criterion for Gaussian data would be

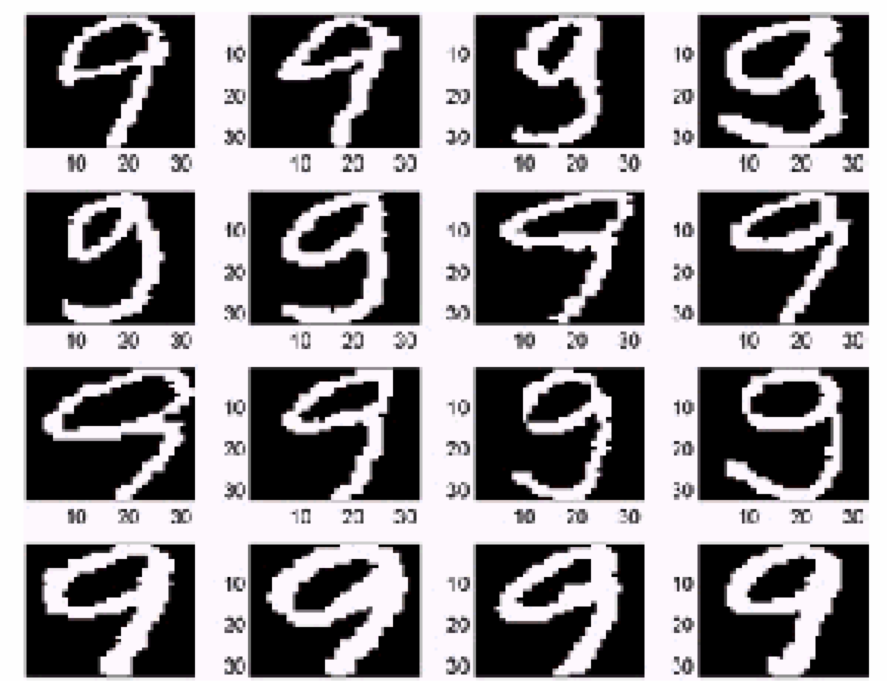
$$J(x,u) = \frac{1}{2}(u - \phi x)^T \, \mathrm{E}\left\{uu^T\right\}^{-1}(u - \phi x)$$

- **Now:** More emphasis on main directions; no invertibility problems!

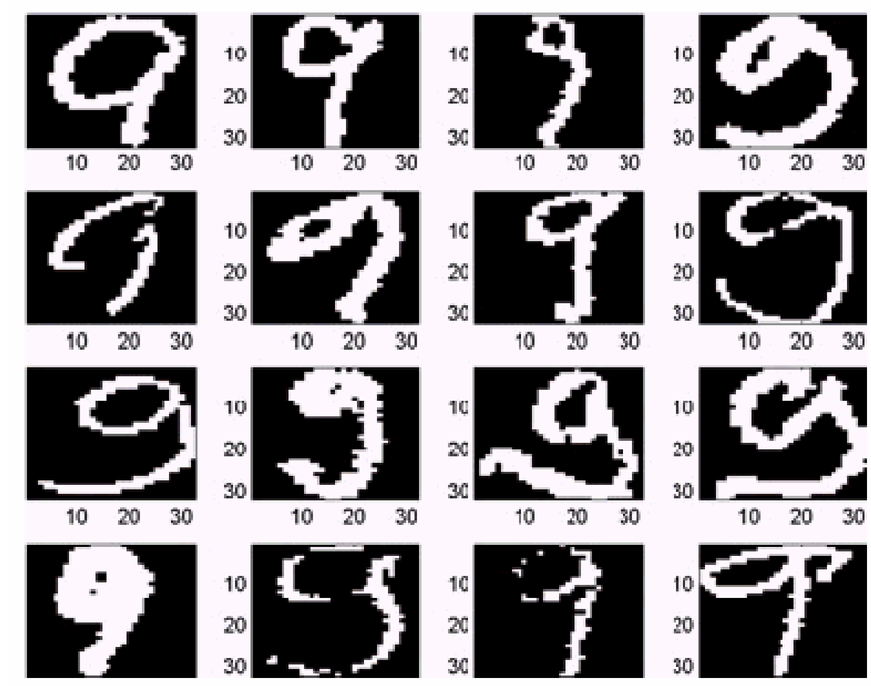
Example: Hand-written digits

- There were a large body of 32x32 pixel images, representing digits from 0 to 9, over 8000 samples (thanks to Jorma Laaksonen)

Examples of typical "9"



Examples of less typical "9"



HELSINKI UNIVERSITY OF TECHNOLOGY
Control Engineering Laboratory
Cybernetics Group

Algorithm for Hebbian/anti-Hebbian learning ...

```
LOOP – iterate for data in kxm matrix U

  % Balance of latent variables
  Xbar = U * (inv(Exx)*Exu)';

  % Model adaptation
  Exu = lambda*Exu + (1-lambda)*Xbar'*U/k;
  Exx = lambda*Exx + (1-lambda)*Xbar'*Xbar/k;

  % PCA rather than PSA
  Exx = tril(ones(n,n)).*Exx;

END

% Recursive algorithm can be boosted with matrix inversion lemma
```

... resulting in Principal Components

- Parameters:

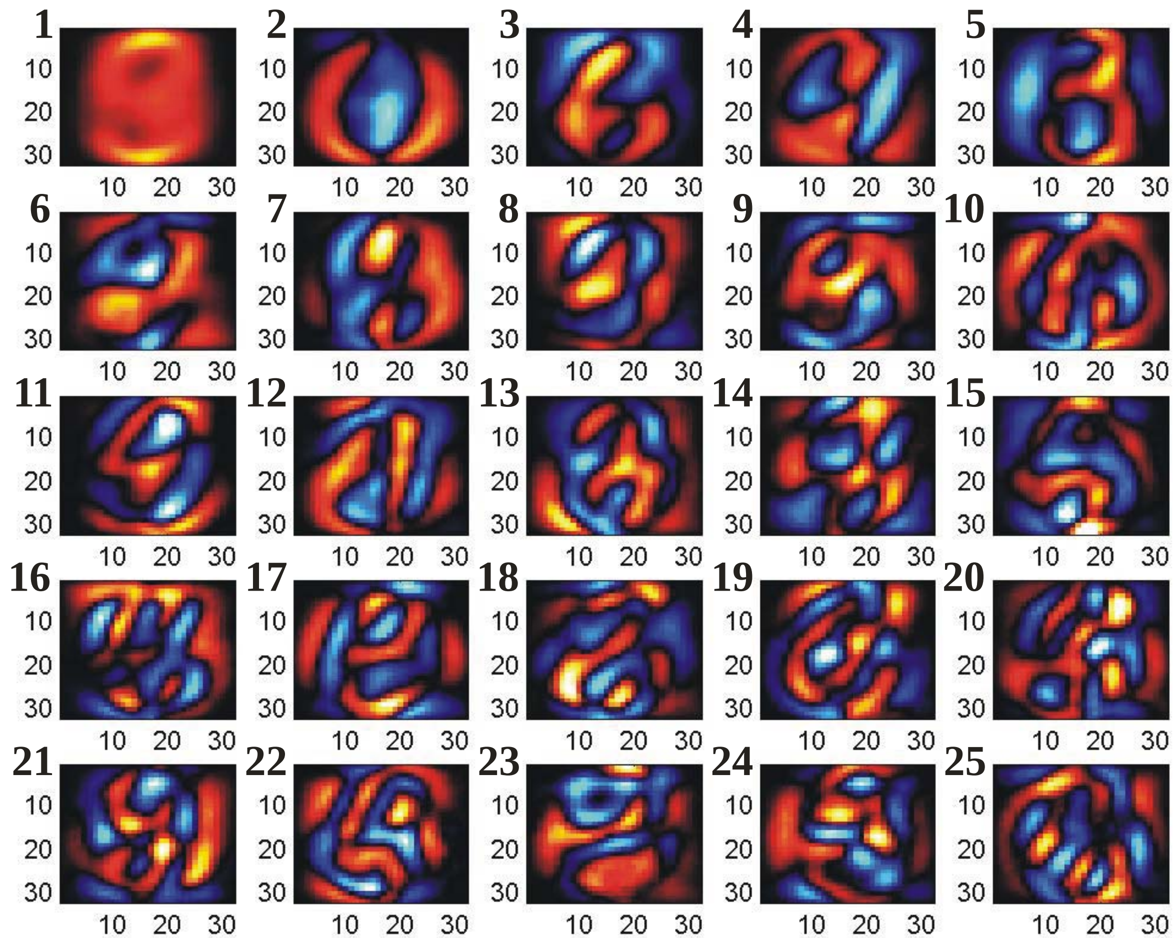
  m = 1024

  n = 16

  $\lambda = 0.5$

DEMO

`digitpca.m`



”Elastic systems”

- New interpretation of cybernetic systems –
- **”First-order cybernetic system”**
  - Finds balance under external pressures, pressures being compensated by internal tensions
  - Any existing (complex) interacting system that maintains its integrity!
  - Implements minimum *observed* deformation energy
- **”Second-order cybernetic system”**
  - Adapts the internal structures to better match the observed environmental pressures – towards *maximum experienced stiffness*
  - Any existing (competing) interacting system that has survived in evolution!
  - Implements minimum *average* observed deformation energy

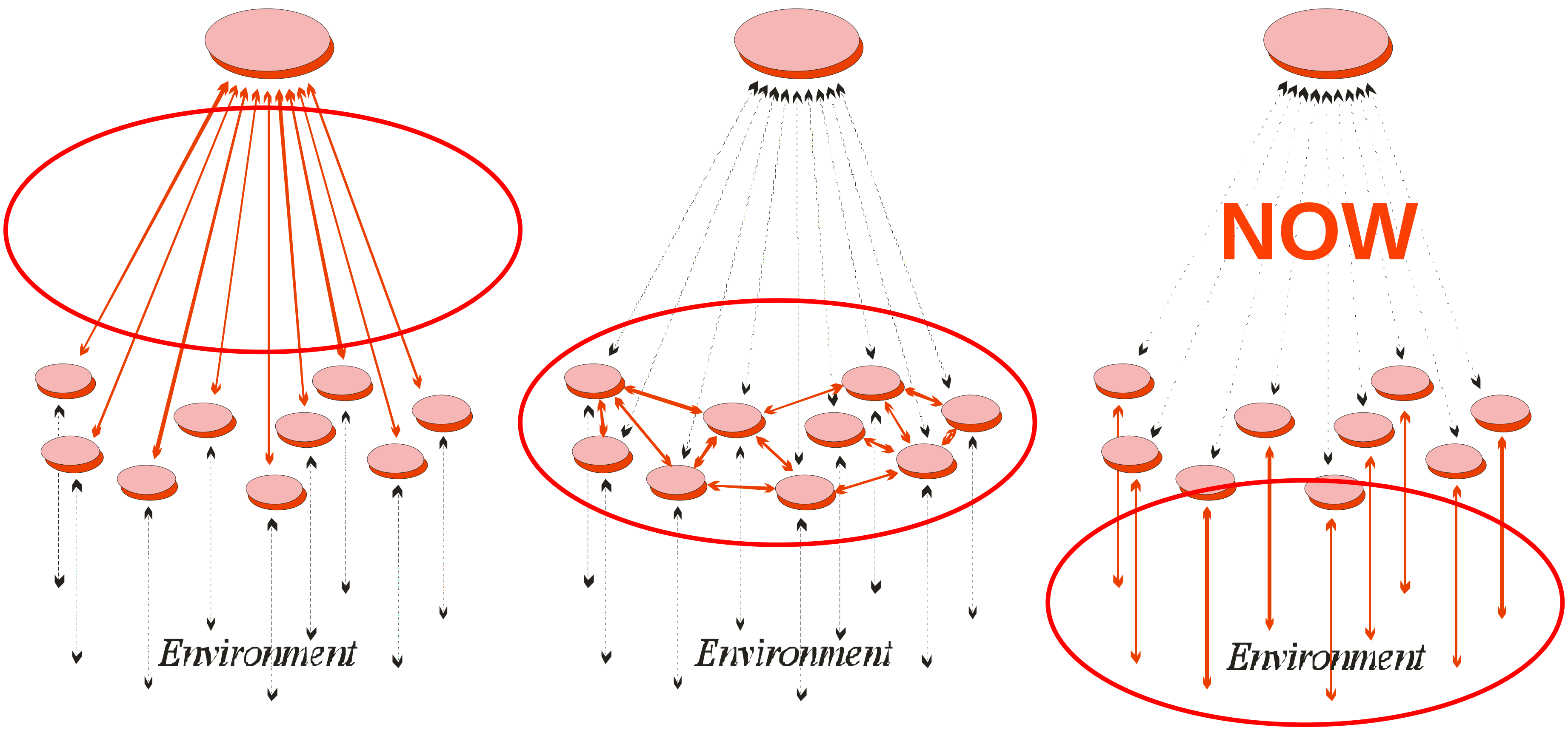
Summary this far

- Emergence in terms of self-regulation (stability) and self-organization (principal subspace analysis) reached
- This is reached applying physiologically plausible operations and model is linear – scalable beyond toy domains
- Learning is local – but not *completely* local: Need ”communication” among neurons (anti-Hebbian structures)
- Roles of signals different: How to motivate the inversion in adaptation direction (anti-Hebbian learning)?

- Solution: Apply non-idealities – in an unorthodox way!
- There exist no unidirectional causal flows in real life systems
- Feedback: Exploiting a signal exhausts that signal

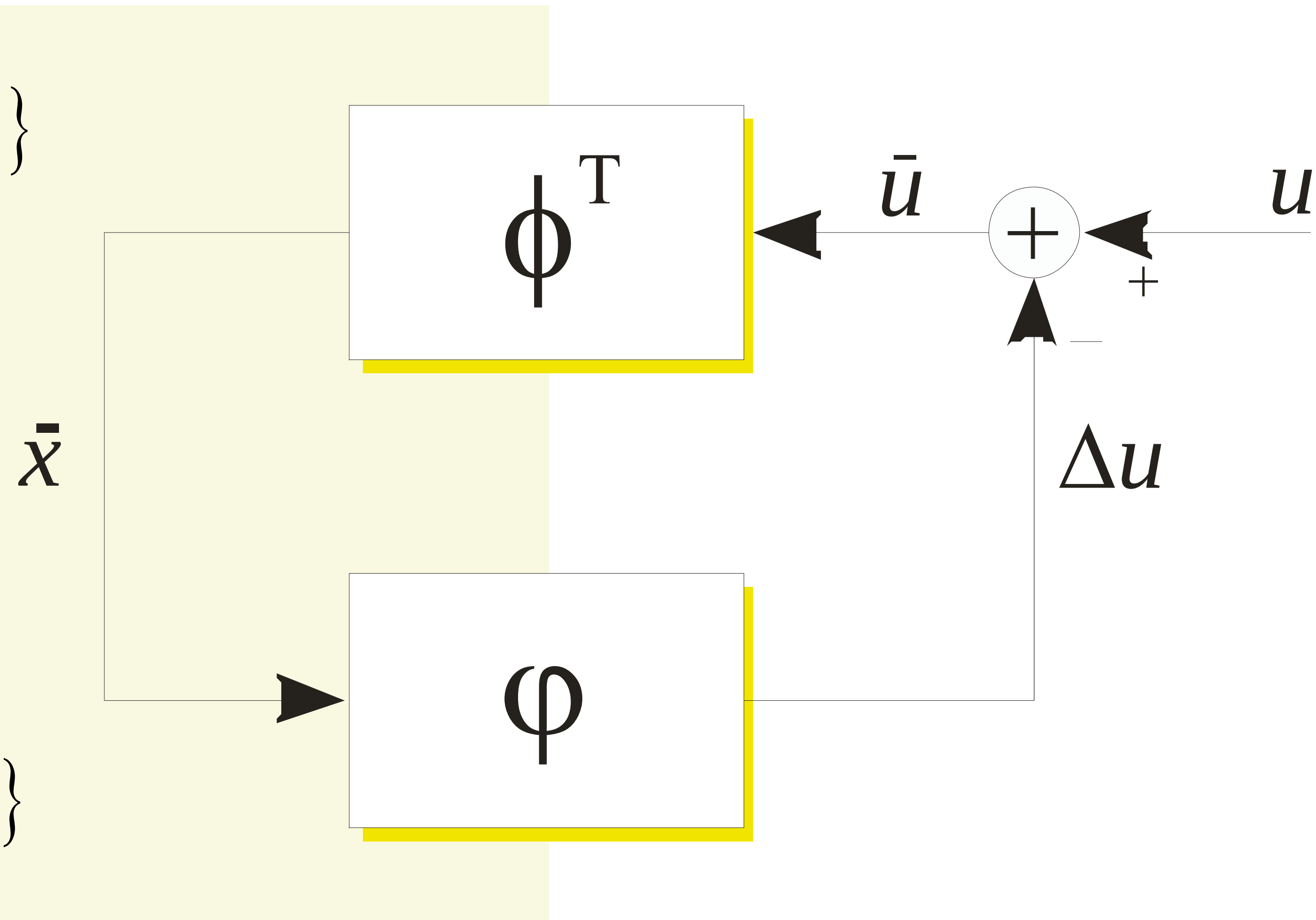
New schema

- Control neither *centralized* nor *distributed* (traditional sense)



Evolutionary balance extended

$$\phi^{\mathrm{T}} = q\,\mathrm{E}\left\{\overline{xu}^{T}\right\}$$



$$\varphi = b\,\mathrm{E}\left\{\overline{ux}^{T}\right\}$$

$$\boxed{\varphi = \frac{b}{q}\phi}$$

Simply ”go for resources”

- Again: Balancing is reached by feedback, but now not explicitly but implicitly through the environment

$$\begin{cases} \bar{x} = \phi^T \bar{u} \\ \bar{u} = u - \varphi \bar{x} \end{cases}$$

- Also environment finds its balance
- Only exploiting locally visible quantities, implement evolutionary adaptation symmetrically as

$$\begin{cases} \phi^T = q \, \mathrm{E}\left\{\overline{x u^T}\right\} \\ \varphi^T = b \, \mathrm{E}\left\{\overline{x u^T}\right\} \end{cases}$$

How to characterize this “environmental balance”?

- Because $\bar{x} = q\mathrm{E}\left\{\overline{xu^T}\right\}\bar{u}$, one can write two covariances:

$$\mathrm{E}\left\{\overline{xu^T}\right\} = q\mathrm{E}\left\{\overline{xu^T}\right\}\mathrm{E}\left\{\overline{uu^T}\right\}$$

and

$$\mathrm{E}\left\{\overline{xx^T}\right\} = q^2\,\mathrm{E}\left\{\overline{xu^T}\right\}\mathrm{E}\left\{\overline{uu^T}\right\}\mathrm{E}\left\{\overline{xu^T}\right\}^T = q\,\mathrm{E}\left\{\overline{xu^T}\right\}\mathrm{E}\left\{\overline{xu^T}\right\}^T$$

so that

$$\begin{cases} I_n = \underbrace{\sqrt{q}\,\mathrm{E}\left\{\overline{xx^T}\right\}^{-1/2}\mathrm{E}\left\{\overline{xu^T}\right\}}_{D^T\theta^T}\ \underbrace{\mathrm{E}\left\{\overline{xu^T}\right\}^T\mathrm{E}\left\{\overline{xx^T}\right\}^{-1/2}\sqrt{q}}_{\theta D} = \theta^T\theta \\ \frac{1}{q}I_n = \underbrace{\sqrt{q}\,\mathrm{E}\left\{\overline{xx^T}\right\}^{-1/2}\mathrm{E}\left\{\overline{xu^T}\right\}}_{D^T\theta^T}\mathrm{E}\left\{\overline{uu^T}\right\}\underbrace{\mathrm{E}\left\{\overline{xu^T}\right\}^T\mathrm{E}\left\{\overline{xx^T}\right\}^{-1/2}\sqrt{q}}_{\theta D} \end{cases}$$

Forget the trivial solution where x_i is identically zero

- Similarly, if $\bar{x} = Q\mathrm{E}\left\{\bar{x}\bar{u}^T\right\}\bar{u}$ for some (diagonal) matrix Q:

$$\mathrm{E}\left\{\overline{xu}^T\right\} = Q\mathrm{E}\left\{\overline{xu}^T\right\}\mathrm{E}\left\{\overline{uu}^T\right\}$$

and

$$\mathrm{E}\left\{\overline{xx}^T\right\} = Q\mathrm{E}\left\{\overline{xu}^T\right\}\mathrm{E}\left\{\overline{uu}^T\right\}\mathrm{E}\left\{\overline{xu}^T\right\}^T Q^T = \mathrm{E}\left\{\overline{xu}^T\right\}\mathrm{E}\left\{\overline{xu}^T\right\}^T Q^T$$

Note: this has to be symmetric, so that

$$\mathrm{E}\left\{\overline{xx}^T\right\} = \mathrm{E}\left\{\overline{xx}^T\right\}^T = Q\mathrm{E}\left\{\overline{xu}^T\right\}\mathrm{E}\left\{\overline{xu}^T\right\}^T$$

Stronger formulation is reached:

$$\theta = \mathrm{E}\left\{\overline{xu}^T\right\}^T \mathrm{E}\left\{\overline{xx}^T\right\}^{-1/2} Q^{1/2}$$

For non-identical q_i, this has to become diagonal also

Equalization of environmental variances

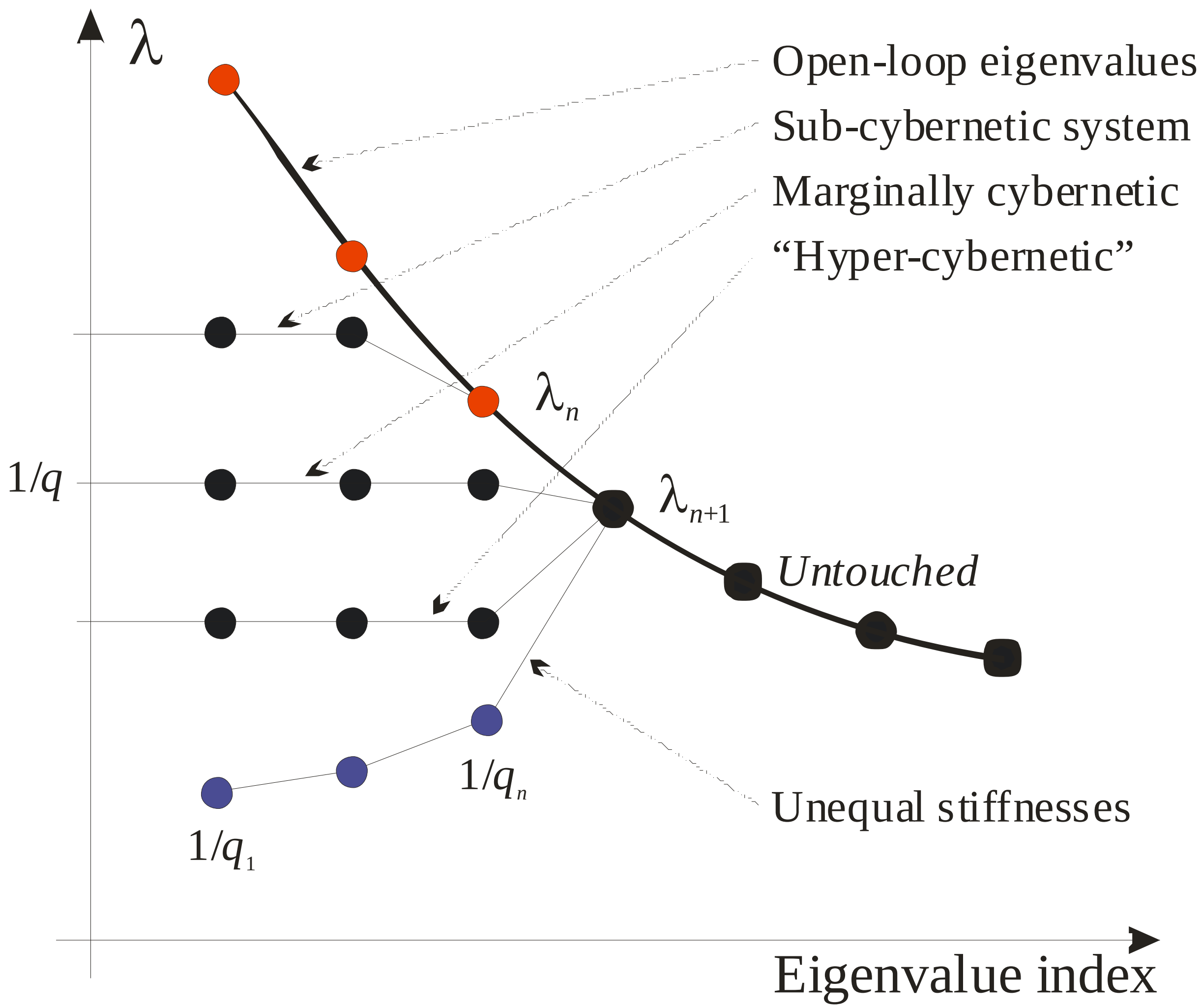
- Because $\theta^T \theta = I_n$ and $\theta^T \mathrm{E}\left\{\overline{uu}^T\right\}\theta = Q^{-1}$, θ consists of the n (most significant) eigenvectors of $\mathrm{E}\left\{\overline{uu}^T\right\}$, and $\mathrm{E}\left\{uu^T\right\}$
- If $n = m$, the variation structure becomes trivial:

$$\mathrm{E}\left\{\overline{uu}^T\right\} = \frac{1}{q} I_m \quad \text{or} \quad \mathrm{E}\left\{\overline{uu}^T\right\} = Q^{-1}$$

- Visible data variation becomes *whitened* by the feedback
- Relation to ICA : Assume that this whitened data is further processed by neurons (FOBI) – but this has to be nonlinear!

- On the other hand, if q_i are different, the modes become separated in the PCA style (rather than PSA)

Still try to avoid nonlinearity!



DEMO

`equalvar.m`

Variance inheritance

- Further – study the relationship between $\bar{x}$ and original u:

$$\overline{x} = \left(I_n + qb\,\mathrm{E}\left\{ \overline{xu}^T \right\} \mathrm{E}\left\{ \overline{xu}^T \right\}^T \right)^{-1} q\mathrm{E}\left\{ \overline{xu}^T \right\} u$$

$$= \left(I_n + b\,\mathrm{E}\left\{ \overline{xx}^T \right\} \right)^{-1} q\mathrm{E}\left\{ \overline{xu}^T \right\} u$$

Multiply from the right by transpose, and take expectations:

$$\left(I_n + b\,\mathrm{E}\left\{ \overline{xx}^T \right\} \right) \mathrm{E}\left\{ \overline{xx}^T \right\} \left(I_n + b\,\mathrm{E}\left\{ \overline{xx}^T \right\} \right)$$

$$= \mathrm{E}\left\{ \overline{xx}^T \right\}^{1/2} \left(I_n + b\,\mathrm{E}\left\{ \overline{xx}^T \right\} \right)^2 \mathrm{E}\left\{ \overline{xx}^T \right\}^{1/2}$$

$$= q^2 \mathrm{E}\left\{ \overline{xu}^T \right\} \mathrm{E}\left\{ uu^T \right\} \mathrm{E}\left\{ \overline{xu}^T \right\}^T$$

$$\left(I_n + b\,\mathrm{E}\left\{\overline{xx}^T\right\}\right)^2$$

$$= q\,\underbrace{\sqrt{q}\mathrm{E}\left\{\overline{xx}^T\right\}^{-1/2}\mathrm{E}\left\{\overline{xu}^T\right\}}_{D^T\theta^T}\mathrm{E}\left\{uu^T\right\}\underbrace{\mathrm{E}\left\{\overline{xu}^T\right\}^T\mathrm{E}\left\{\overline{xx}^T\right\}^{-1/2}\sqrt{q}}_{\theta D}$$

Solving for the latent covariance:

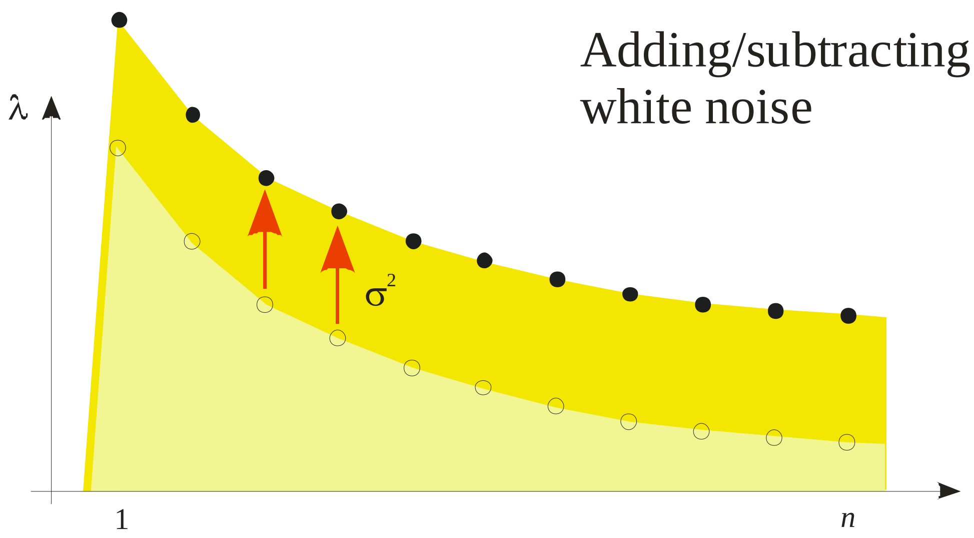
$$\mathrm{E}\left\{\overline{xx}^T\right\} = \frac{1}{b}\left(q\,D^T\theta^T\mathrm{E}\left\{uu^T\right\}\theta D\right)^{1/2} - \frac{1}{b}I_n$$

This means that the external and internal eigenvalues (variances) are related as follows:

$$\frac{\sqrt{q_i\lambda_i}-1}{b_i}$$

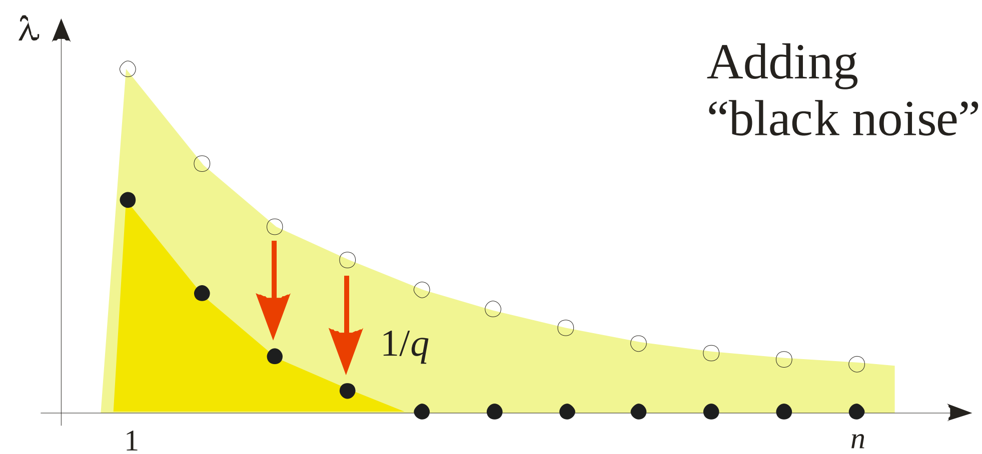
– There must hold $q_i\lambda_i > 1$

Effect of feedback = add ”black noise”

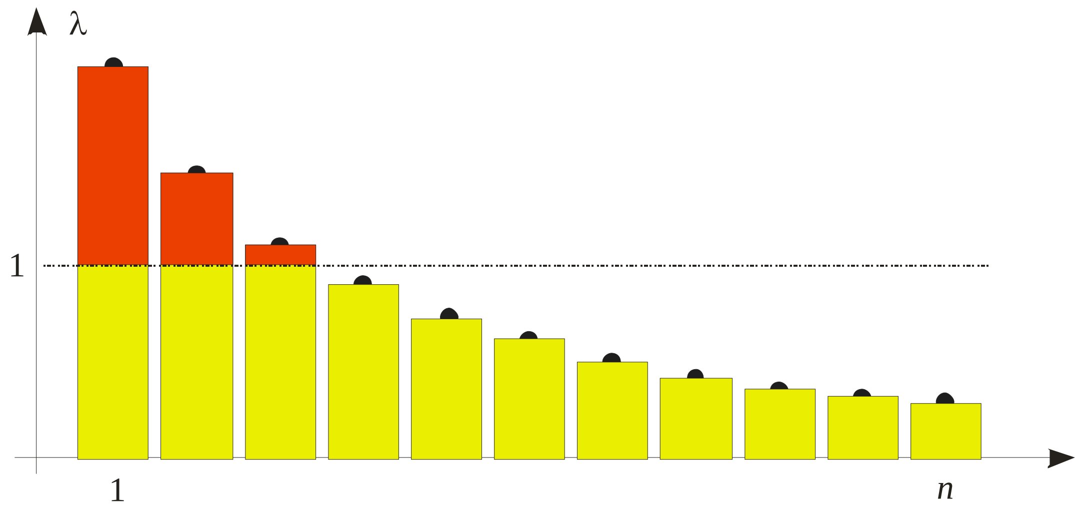


- White noise = Constant increase in all directions
- ”Black noise” = Decrease in all directions (if poss.)

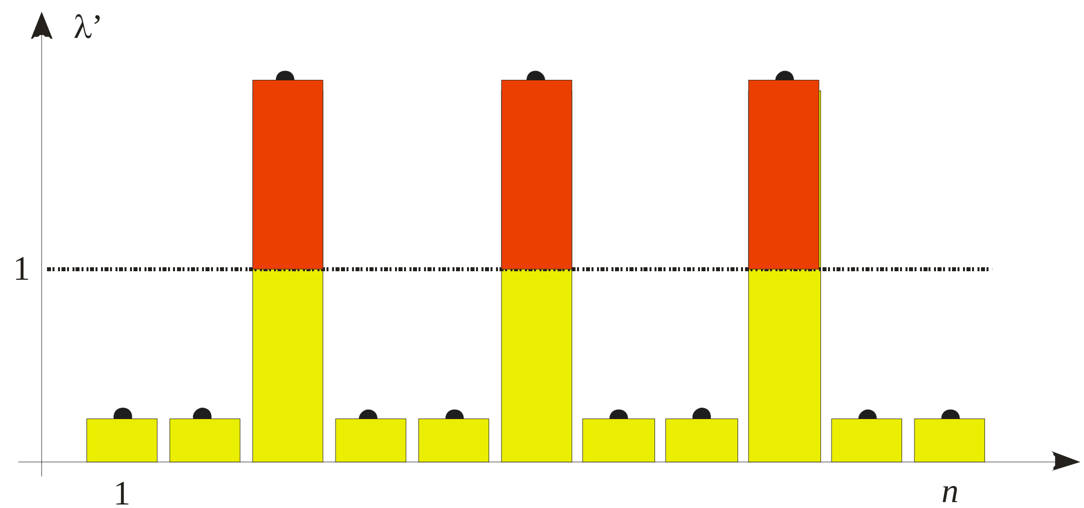
If $q_i = \lambda_i$ and $b_i = 1$, variances are $\lambda_i - 1$



Results of orthogonal basis rotations



Mathematical view of data



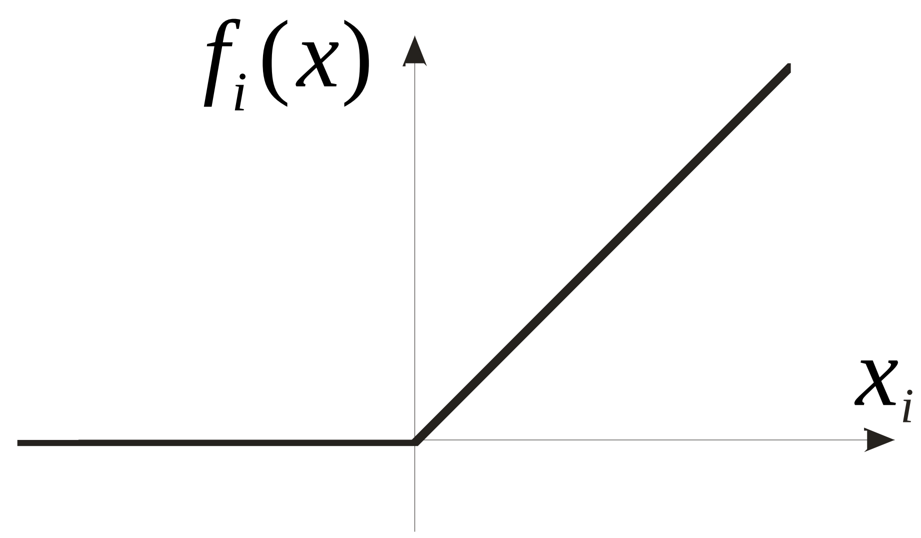
Cybernetic view of data

- Total variance above zero level intact regardless of the rotations
- Total variance above 1 changes!

Towards differentiation of features

- A simple example of nonlinear extensions: CUT function
- If variable is positive, let it through; otherwise, filter it out – Well in line with modeling of activity in neuronal systems:
  - Frequencies cannot become negative (interpretation in terms of pulse trains)
  - Concentrations cannot become negative (interpretation in terms of chemicals)
- Makes modes separated
- Still: *End result linear!*

$$f_i(x) = \begin{cases} x_i, & \text{when } x_i > 0 \\ 0, & \text{when } x_i \leq 0 \end{cases}$$



Algorithm for Hebbian feedback learning ...

```
LOOP – iterate for data in kxm matrix U

  % Balance of latent variables
  Xbar = U * (inv(eye(n)+q*Exu*Exu')*q*Exu)';

  % Enhance model convergence by nonlinearity
  Xbar = Xbar.*(Xbar>0);

  % Balance of the environmental signals
  Ubar = U - Xbar*Exu;

  % Model adaptation
  Exu = lambda*Exu + (1-lambda)*Xbar'*Ubar/k;

  % Maintaining system activity
  Exx = Xbar'*Xbar/k;
  q = q + P*diag(ref - sqrt(diag(Exx)));

END
```

... resulting in *Sparse Components!*

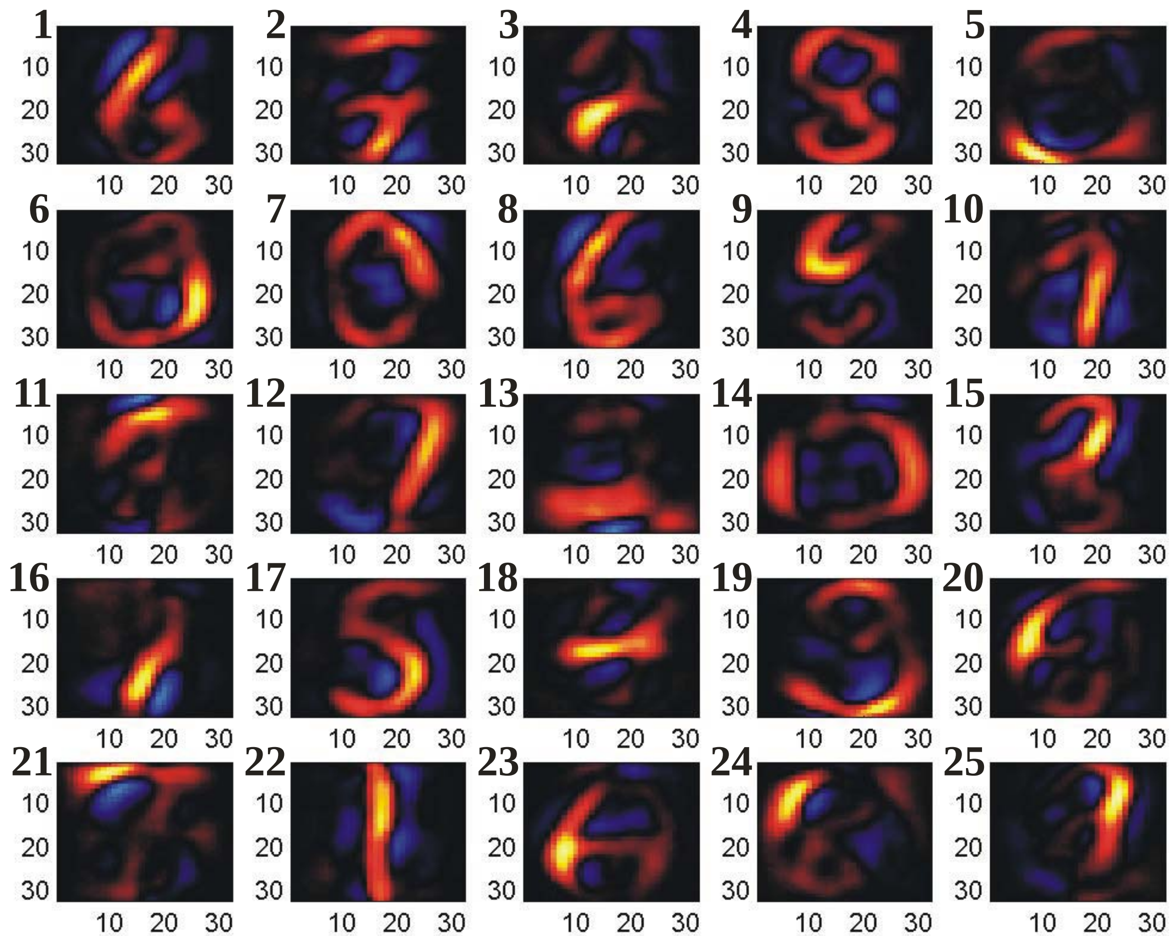
- Parameters:

  $m = 1024$

  $n = 16$

  $\lambda = 0.97$

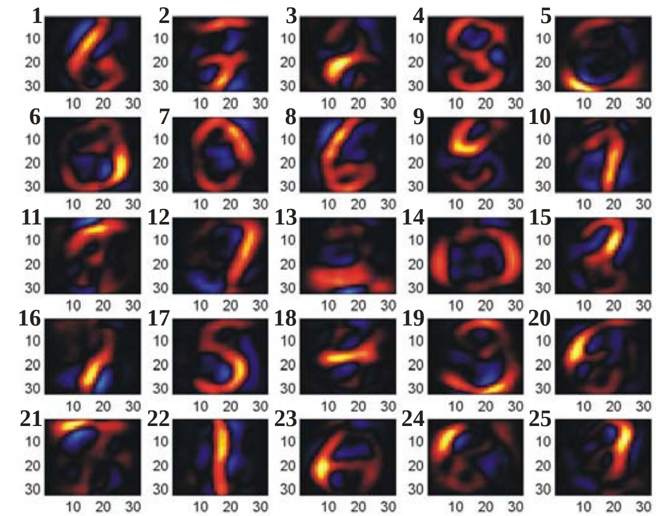
  ref = 1

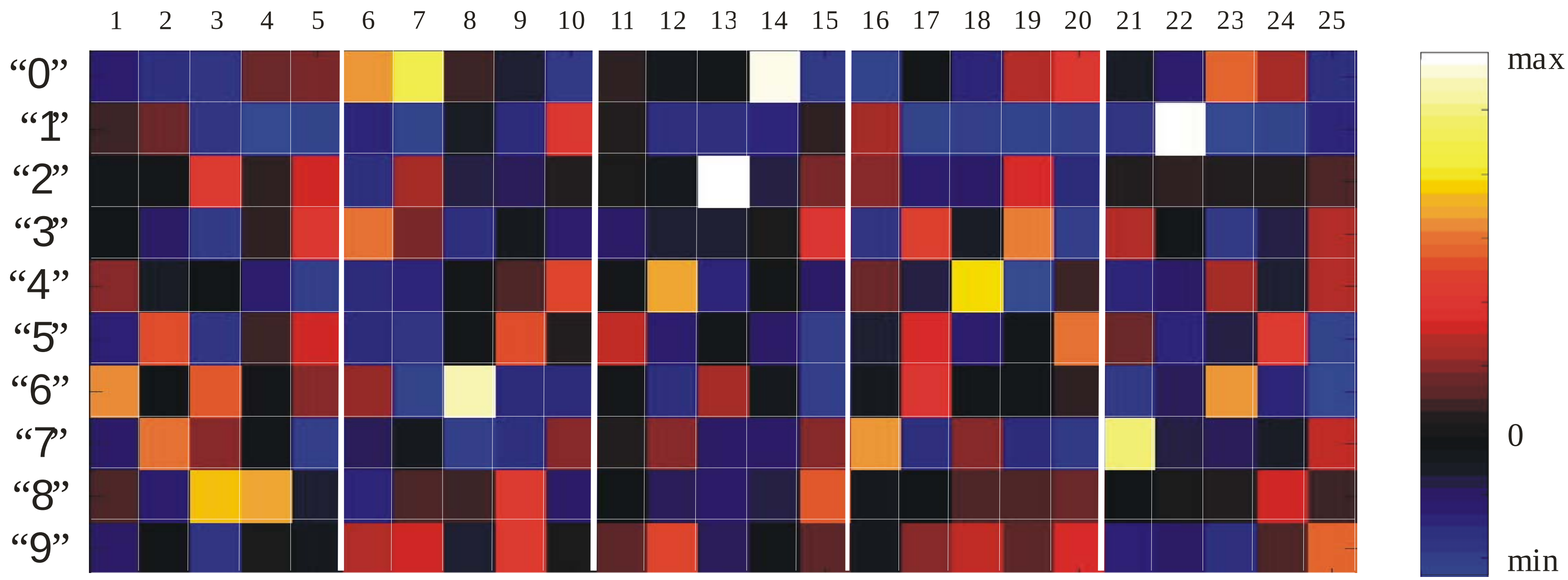


DEMO

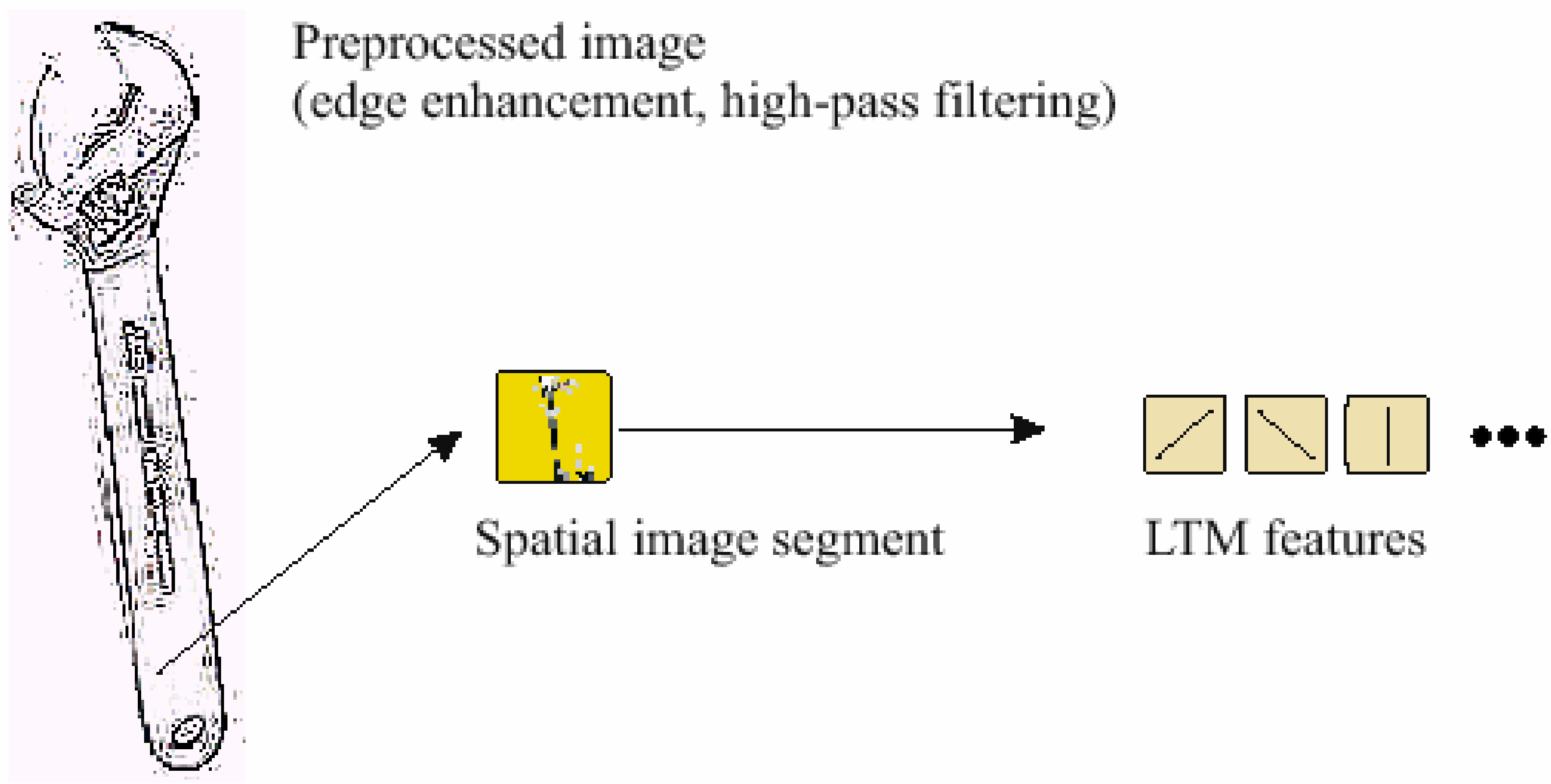
`digitfeat.m`

- ”Work load” becomes distributed
- Correlations between inputs and neuronal activities shown below:



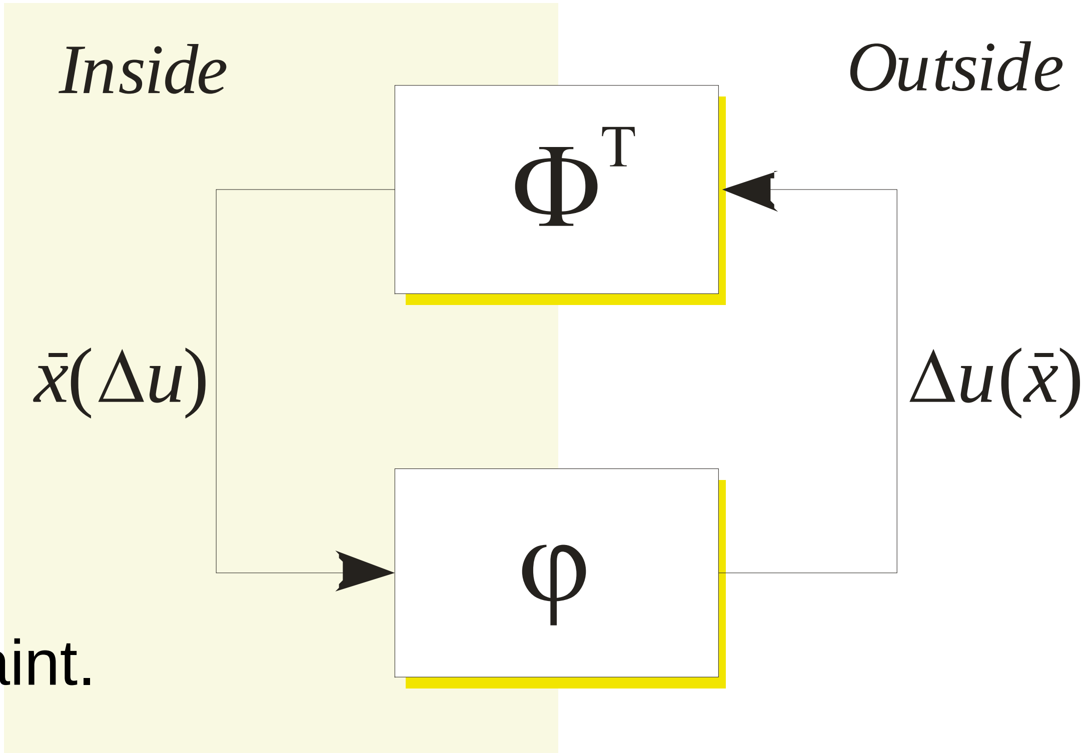


- Visual V1 cortex seems to do this kind of decomposing



"Loop invariant"

- There are two main structures that dictate the properties of the Hebbian feedback system from different points of view:
  - Hebbian learning (studied above)
  - Feedback (studied now):



- It must be so that

$$\bar{x} = \Phi^T \varphi \, \bar{x}$$

or

$$\Phi^T \varphi = I_n$$

- This is a harsh constraint.

Mapping in terms of data

- Study how the feedback mapping can be characterized. Because

$$\Delta u = \varphi \overline{x}$$

there holds

$$\mathrm{E}\left\{\Delta u \overline{x}^T\right\} = \varphi\, \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}$$

or, when manipulating,

$$\varphi = \mathrm{E}\left\{\Delta u \overline{x}^T\right\} \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^{-1} = \mathrm{E}\left\{\overline{x}\Delta u^T\right\}^T \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^{-1}$$

Note:
Least-squares fitting formula!

- It turns out: To obey $\Phi^T \varphi = I_n$ feedforward mapping is

$$\Phi^T = \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^{-1} \mathrm{E}\left\{\overline{x}\Delta u^T\right\}$$

The same derivations – for Δu now

- Again – derive the statistical model of balances $\bar{x}(\Delta u)$
- Assume that dynamics of u is essentially slower than that of x and study the covariance properties:

$$\mathrm{E}\left\{\overline{x}\overline{x}^T\right\} = \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^{-1} \mathrm{E}\left\{\overline{x}\Delta u^T\right\} \mathrm{E}\left\{\Delta u \Delta u^T\right\} \mathrm{E}\left\{\overline{x}\Delta u^T\right\}^T \mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^{-1}$$

or

$$\mathrm{E}\left\{\overline{x}\overline{x}^T\right\}^3 = \mathrm{E}\left\{\overline{x}\Delta u^T\right\} \mathrm{E}\left\{\Delta u \Delta u^T\right\} \mathrm{E}\left\{\overline{x}\Delta u^T\right\}^T$$

or

$$\left(\Phi^T \mathrm{E}\left\{\Delta u \Delta u^T\right\} \Phi\right)^3 = \Phi^T \mathrm{E}\left\{\Delta u \Delta u^T\right\}^3 \Phi \qquad n < m$$

- Same PSA properties – now for signals $\bar{x}$ and Δu

- The mapping matrix Φ also spans the subspace determined by φ ...
- Trivial result if no adaptation (however, note nonlinearity!)
- But combined with the Hebbian learning, the mapping matrices adapt to represent the principal subspace of u

  (Note that this all applies only if there holds $x \not\equiv 0$)

- There are also more fundamental consequences ...
- Conclusion: Essentially the system is modeling *its own behavior* in the environment, or mapping between $\bar{x}$ and Δu
- One can see $\mathrm{E}\left\{\bar{x}_i \Delta u_j\right\}$ as an atom of causal information

Towards modeling of causality

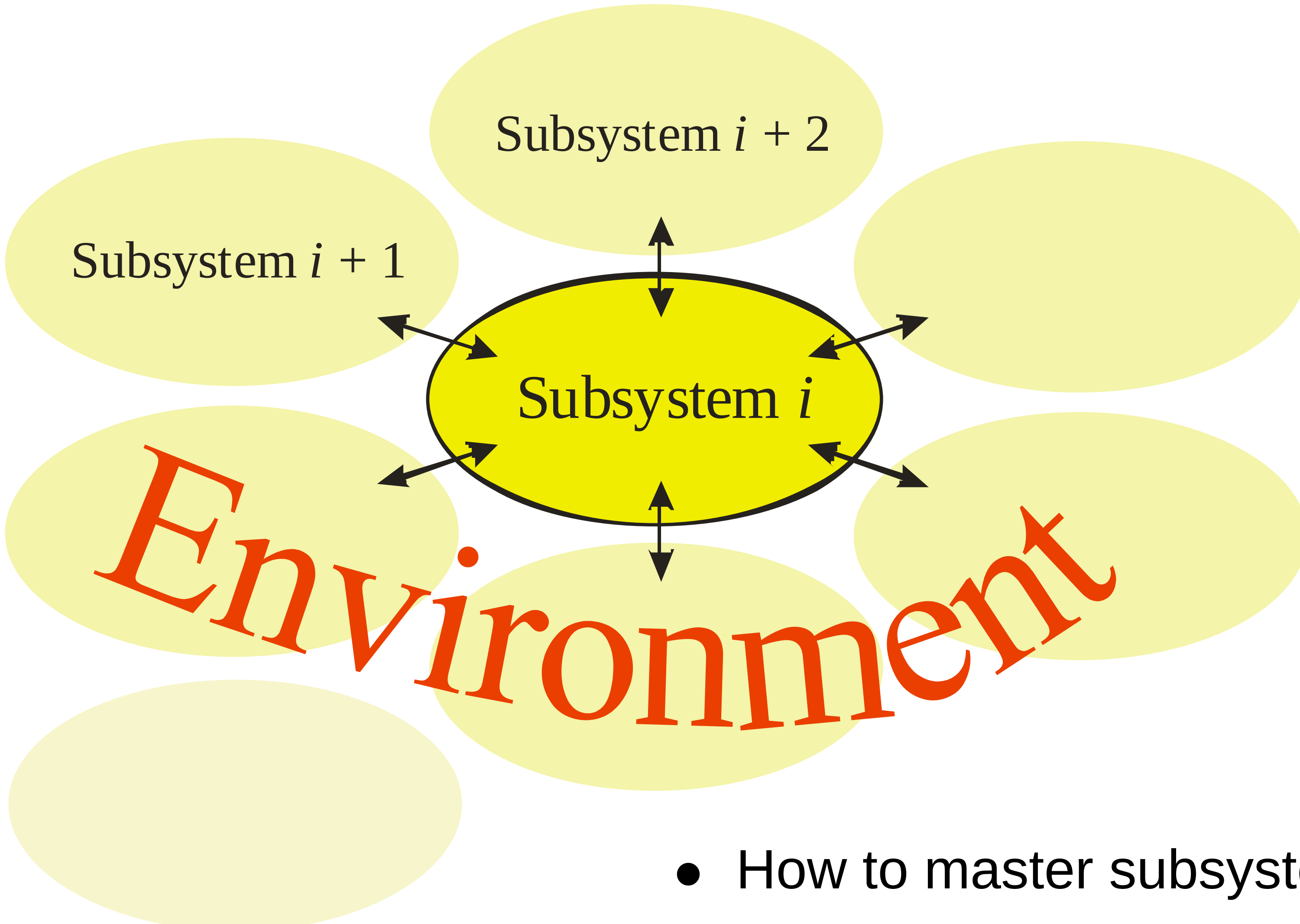
- Age-old observation (Hume): One can only see correlations in data, not causalities
- Another observation (Kant): Human still *for some reason* is capable of constructring causal models

- Hebbian feedback learning:

  Modeling of *results of own actions* in the environment (actions being *reactions* to phenomena in the environment)
- Now one implicitly knows what is cause, what is effect

- Learning needs to be of ”hands-on” type, otherwise learning (applying explicit anti-Hebbian law) becomes superficial?!

Yet another elasticity benefit

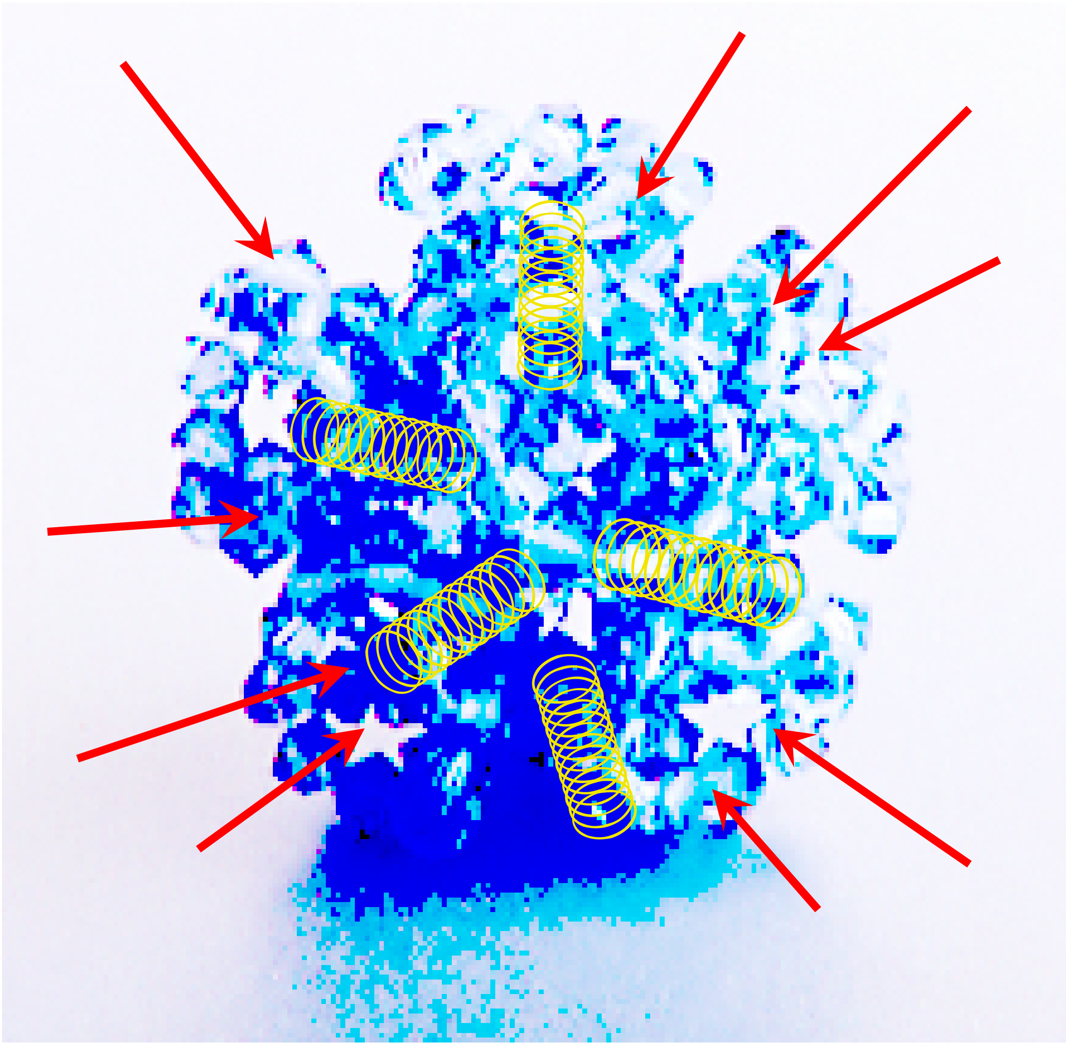


- How to master subsystems?

Analogues rehabilitated

- When applying linearity, the number of available structures is rather limited – there are more systems than models!
- This idea has been applied routinely: Complicated systems are visualized in terms of structures with the same dynamics
- In the presence of modern simulation tools, this kind of lumped parameter simplifications seem rather outdated ...
- However, in the case of really complicated distributed parameter systems, mechanical analogues may have reincarnation – steel plates are still simple to visualize!
- Another class of analogues (current/voltage rather than force/deformation) can also be constructed:
  - External forces are the loads; the deformation is the voltage drop, and the control action is the increased current

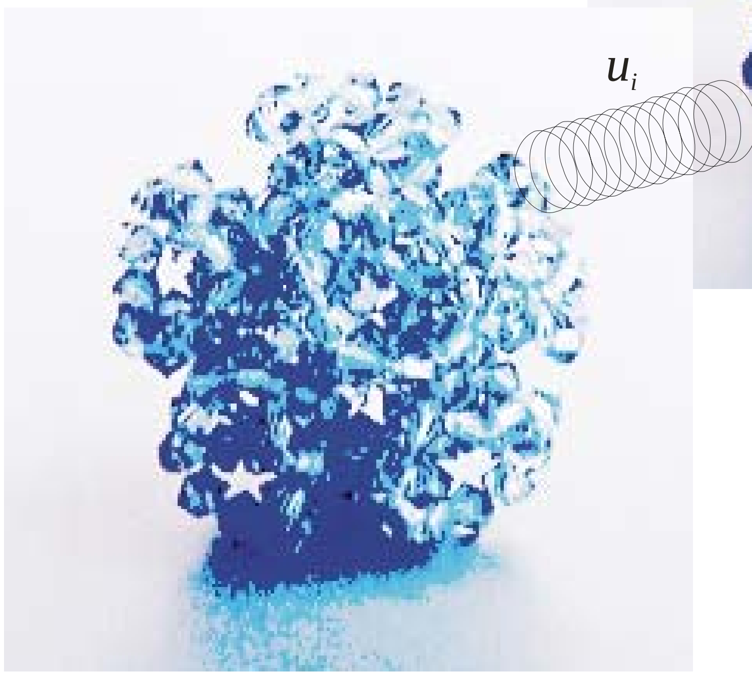
For mechanical engineers ...

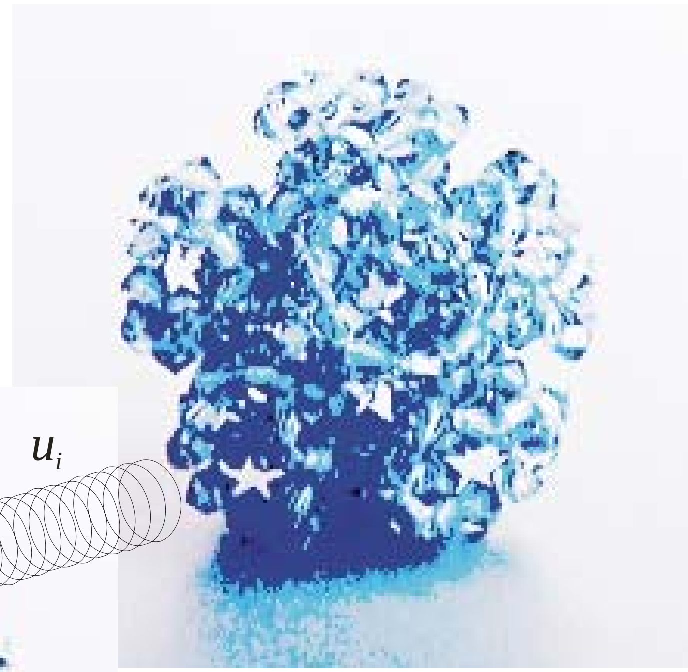


- Assumption: Internal tensions compensate external pressures
- The forces acting on the system cannot all be captured, nor can the interconnections among actions
- *The complexity of the system/environment is projected onto the finite, discrete set of concrete actors*

... and for electrical engineers

Cybernetic domain #1





Cybernetic domain #2

- For maximum energy transfer impedances have to match!

Cybernetics Rules!

... But what are those rules?

Let us find it out!

http://www.control.hut.fi/cybernetics