Session 10

Self-Similarity and Power Laws

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The aim of this chapter is to present the relate the common features of complex systems, self-similarity and self-organization to power (scaling) laws and to fractal dimension and to show how all these are intertwined together. First, the basic concepts of self-similarity, self-organization, power laws and fractals are represented. Then, the power laws and fractals are discussed more detailed. Finally, examples of the applications of power laws and fractal dimension are demonstrated.

10.1 Introduction

The *power laws* and *fractal dimensions* are just two sides of a coin and they have a tight relationship joining them the together, as shown in Fig. 10.1. The relationships can be clarified with a mathematical discussion. The general equation for power law is shown in (10.1). It is a mathematical pattern in which the frequency of an occurrence of a given size is inversely proportionational to some power n of its size:

$$y(x) = x^{-n}. \tag{10.1}$$

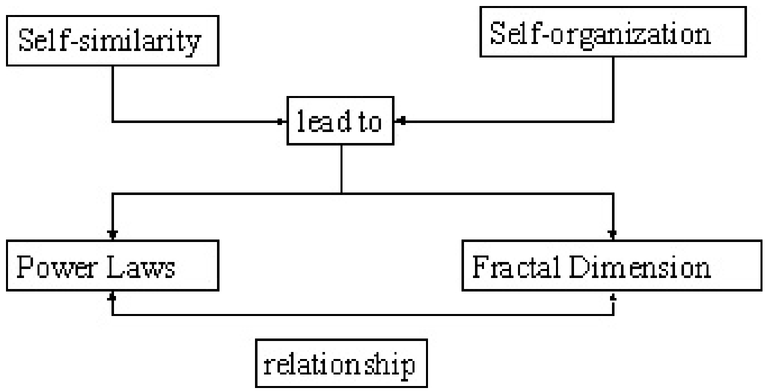

Figure 10.1: Relationships between self-similarity, self-organization, power laws and fractal dimension

Note that

$$y(\lambda x) = (\lambda x)^{-n} = \lambda^{-n} x^{-n} = \lambda^{-n} y(x). \tag{10.2}$$

It turns out that the power law can be expressed in "linear form" using logarithms:

$$\log(y(x)) = -n \log(x), \tag{10.3}$$

where the coefficient n represents the *fractal dimension* [2].

The mathematical relationship connecting self-similarity to power laws and to fractal dimension is the *scaling equation.* For an self-similar observable $A(x)$, which is a function of a variable x, a scaling relationship holds:

$$A(\lambda x) = \lambda^s A(x), \tag{10.4}$$

where λ is a constant factor and s is the scaling exponent, which is independent of x. Looking at (10.2), it is clear that the power law obeys the scaling relationship.

The data emerging from the combination of self-similarity and self-organization can not be described by either Normal or exponential distribution.The reason is, that emergence of order in complex systems is fundamentally based on correlations between different levels of scale. The organization of phenomena that belong at each level in the hierarchy rules out a preferred scale or dimension. The relationships in this type of systems are best described by power laws and fractal dimension [3].

10.2 Self-Similarity

Self-similarity means that a structure, or a process, and a part of it appear to be the same when compared. A self-similar structure is infinite and it is not differentiable in any point.

Approximate self-similarity means that the object does not display perfect self-similarity. For example a coastline is a self-similar object, a natural fractal, but it does not have perfect self-similarity. A map of a coastline consists of bays and headlands, but when magnified, the coastline is not identical but statistically the average proportions of bays and headlands remains the same no matter the scale [4].

It is not only natural fractals that display approximate self-similarity; the Mandelbrot set is another example. Identical pictures do not appear straight away, but when magnified, smaller examples will appear at all levels of magnification [4].

Statistical self-similarity means that the degree of complexity repeats at different scales instead of geometric patterns. Many natural objects are statistically self-similar where as artificial fractals are geometrically self-similar [5].

Geometrical similarity is a property of the space-time metric, whereas physical similarity is a property of the matter fields. The classical shapes of geometry do not have this property; a circle if on a large enough scale will look like a straight line. This is why people believed that the world was a flat pancake, the earth just looks that way to humans [4,6].

Examples of self-similar systems

One well-known example of self-similarity and scale invariance is fractals, patterns that form of smaller objects that look the same when magnified.

Many natural forms, such as coastlines, fault and joint systems, folds, layering, topographic features, turbulent water flows, drainage patterns, clouds, trees, leaves, bacteria cultures [4], blood vessels, broccoli, roots, lungs and even universe, etc., look alike on many scales [6]. It appears as if the underlying forces that produce the network of rivers, creeks, streams and rivulets are the same at all scales, which results in the smaller parts and the larger parts looking alike, and these looking like the whole. [3,4]

"Human-made" self-similar systems include for example music [3], behavior of ethernet traffic, programming languages [7], architecture (of asian temples etc.). The process of human cognition facilitates scales and similarity. Human mind groups similar objects of the same size into a single level of scale. This process has been compared with digital image compression, because it reduces the amount of presented information by a complex structure [3]. Self-similarity in music comes from the coherent nature of the sounds. The coherencies are agreeing with each other in every scale and dimension which they are perceived. [5]. The expansion of the Universe from the big bang and the collapse of a star to a singularity might both tend to self-similarity in some circumstances [6].

A self-similar program is a program that can mutate itself into a new, more complex program, that is also self-similar. For example, a self-similar language can be extended with a new language feature, which will be used to parse or compile the rest of the program. Many languages can be made self-similar. When language is simple but powerful (Lisp, SmallTalk), self-similarity is easier to achieve than when it is complex but powerful (Java, C++) [7].

10.2.1 Self-organization

Two types of stable systems can be found in the physical universe: the death state of perfect equilibrium and the infinitely fertile condition of self-organized non-equilibrium. Self-organization provides useful models for many complex features of the natural world, which are characterized by fractal geometries, cooperative behavior, self-similarity of structures, and power law distributions. Openness to the environment and coherent behavior are necessary conditions for self-organization and the growth of complexity [8].

Because of a common conceptual framework or microphysics, self-organizing systems are characterized by self-similarity and fractal geometries, in which similar patterns are repeated with different sizes or time scales without changing their essential meaning. Similar geometric patterns are repeated at different sizes and are expressive of a fundamental unity of the system such as braiding patterns ranging from streambeds to root systems and the human lung [8].

Systems as diverse as metabolic networks or the world wide web are best described as networks with complex topology. A common property of many large networks is that the vertex connectivities follow a scale-free power-law distribution. This feature is a consequence of two generic mechanisms shared by many networks: Networks expand continuously by the addition of new vertices, and new vertices attach preferentially to already well connected sites.

A model based on these two ingredients reproduces the observed stationary scale-free distributions, indicating that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of the individual systems [9].

10.2.2 Power laws

Power law is one of the common signatures of a nonlinear dynamical process, i.e., a chaotic process, which is at a point self-organized. With power laws it is possible to express self-similarity of the large and small, i.e., to unite different sizes and lengths. In fractals, for example, there are many more small structures than large ones. Their respective numbers are represented by a power law distribution. A common power law for all sizes demonstrates the internal self-consistency of the fractal and its unity across all boundaries. The power law distributions result from a commonality of laws and processes at all scales [2].

The scaling relationship of power laws applies widely and brings into focus one important feature of the systems considered. In general it is not evident that major perturbation will have the larger effect and a minor one only a small effect. The knock-on effect of any perturbation of a system can vary from zero to infinite — there is an inherent fractal unpredictability [10].

When using the power laws, one must notice that statistical data for a phenomenon that obeys one of the power laws (exponential growth) is biased towards the lower part of the range, whereas that for a phenomenon with saturation (logarithmic growth) tends to be biased towards the upper part of the range [11].

Applications of power laws

The natural world is full of power law distributions between the large and small: Earthquakes, words of the English language, interplanetary debris, and coastlines of continents. For example power laws define the distribution of catastrophic events in Self-Organized Critical systems. If a SOC system shows a power law distribution, it could be a sign that the system is at the edge of chaos, i.e., going from a stable state to a chaotic state. With power laws it is be useful to predict the phase of this type of systems. A power law distribution is also a litmus test for self-organization, self-similarity and fractal geometries [2,12].

The Gutenberg-Richter law, used in seismography, is an empirical observation that earth quakes occur with a power law distribution. The crust of the Earth, buckling under the stress of plates sliding around, produces earthquakes. Every day at some parts of the earth occur small earth quakes, that are too weak to detect without instruments. A little more bigger earthquakes, that rattle dishes are less common and the big earth quakes, that cause mass destructions happen only once in a while [13].

Power laws are applied to monitor the the acoustic emissions in materials which are used for bridges etc. Internal defects in materials make popping sounds, acoustic emissions, under stress. Engineering materials contain tiny cracks that grow under stress — until they grow large enough to cause materials to fail. This can cause the failure of buildings, bridges and other societal structures. One way to determine the condition of structures is to monitor the acoustic emissions. This monitoring method is simple and inexpensive form of non-destructive testing. With the help of the method it is also possible to develop new techniques to make better materials and design structures less vulnerable to cracking [13].

Matthew-effect on scientific communities is a discovery, that the relationship between the amount of citations received by members of a scientific community and their publishing size follows power law with exponent 1,27±0,0 . The exponent is shown to be constant with time and relatively independent of the nationality and size of a science system. Also the publishing size and the size rank of the publishing community in a science system have a power-law relationship. Katz has determined the exponent to be –0,44±0.01. The exponent should be relatively independent of the nationality and size of the science system although, according to Katz, the rank of a specific community in a science system can be quite unpredictable [14].

Some examples of power laws in biology and sociology are the laws of Kleiber, which relates metabolic rate to body mass in animals; Taylor's power law of population fluctuations and Yoda's thinning law, which relates density of stems to plant size [15].

In the following subchapters concentration is placed on Zipf's and Benford's laws. Typical for the phenomena is that large occasions are fairly rare, whereas smaller ones are much more frequent, and in between are cascades of different sizes and frequencies. With Zipf's law the structure of the phenomena can be explained and it is possible to do some sort of prediction for the earth quakes coming in the future.

10.2.3 Zipf's law

Zipf's law is named after the Harvard professor George Kingsley Zipf (1902–1950). The law is one of the scaling laws and it defines that the frequency of occurrence of some event (P), as a function of the rank (i) is a power-law function [16,17].

In Zipf's law the quantity under study is inversely proportional to the rank. The rank is determined by the frequency of occurrence and the exponent a is close to unity [17].

Zipf's law describes both common and rare events. If an event is number 1 because it is most popular, Zipf's plot describes the common events (e.g., the use of English words). On the other hand, if an event is number 1 because it is unusual (biggest, highest, largest, ...), then it describes the rare events (e.g., city population) [17].

Benoit Mandelbrot has shown that a more general power law is obtained by adding constant terms to the denominator and power. In general, denominator is the rank plus first constant c and power is one plus a second constant d. Zipf's law is then the special case in which the two constants, c and d are zero [16]:

$$P_i \sim 1/(i+c)^{(1+d)}. \tag{10.5}$$

For example the law for squares can be expressed with $d = 1$, so the power will be $a = 2$ [16]:

$$P_i \sim 1/(i+c)^2. \tag{10.6}$$

Mathematics gives a meaning to intermediate values of the power as well, such as $3/4$ or 1.0237 [16].

Mandelbrot's generalization of Zipf's law is still very simple: The additional complexity lies only in the introduction of the two new adjustable constants, a number added to the rank and a number added to the power 1, so the modified power law has two additional parameters.

There is widespread evidence that population and other socio-economic activities at different scales are distributed according to the Zipf's law and that such scaling distributions are associated with systems that have matured or grown to a steady state where their growth rates do not depend upon scale [18].

Applications of Zipf's law

The constants of the generalized power law or modified Zipf's law should be adjusted to gain the optimal fit of the data. The example of the cities in the USA will clarify this. In the Table 10.1 cities are listed by their size (rank). The population of the cities year 1990 are compared to the predictions of unmodified and modified Zipf's law. The unmodified Zipf's law uses 10 milloin people as the coefficient for the equation 1.

$$P_i = 10,000,000 \cdot 1/i^1. \qquad (10.7)$$

The modified Zipf's law consists of population coefficient 5 million and constants $c = -2/5$ and $d = -1/4$.

$$P_i = 5,000,000 \cdot 1/(i - 2/5)^{(3/4)}. \qquad (10.8)$$

The sizes of the populations given by unmodified Zipf's law differ from the real sizes approximately 30 % and the sizes by modified Zipf's law only about 10 %. The obvious conclusion is that the modified version of Zipf's law is more accurate than the original one. Although the predictions of the original one are quite good too [16].

Other famous examples of Zipf's law are: The frequency of English words and the income or revenue of a company. The income of a company as a function of the rank should also be called the Pareto's law because Pareto observed this at the end of the last century [17,19].

The english word example is illustrated by counting the top 50 words in 423 TIME magazine articles (total 245,412 occurrences of words), with "the" as the number one (appearing 15861 times), "of" as number two (appearing 7239 times), "to" as the number three (6331 times), etc. When the number of occurrences is plotted as the function of the rank (1, 2, 3, etc.), the functional form is a power law function with exponent close to 1 [17].

Shiode and Batty have proposed a hypothesis of the applicability of Zipf's law for the growth of WWW domains. Most of the mature domains follow closely Zipf's law but the new domains, mostly in developing countries, do not. The writers argue that as the WWW develops, all domains will ultimately follow the same power laws as these technologies mature and adoption becomes more uniform. The basis for this hypothesis was the observation that the structure in the cross-sectional data, which the writers had collected, was consistent with a system that was rapidly changing and had not yet reached its steady state [18].

Table 10.1: Zipf's Law applied to population of cities in the USA. Unmodified Zipf's law uses population size of 10 million and constants 0. Modified Zipf's law uses population size of 5 million and constants -2/5 and -1/4 [16].

Rank (n)	City	Population (1990)	Unmodified Zipf's law	Modified Zipf's Law
1	NewYork	7.322.564	10.000.000	7334.265
7	Detroit	1.027.974	1.428.571	1214.261
13	Baltimore	736.014	769.231	747.639
19	Washington, D.C.	606.900	526.316	558.258
25	New Orleans	496.938	400.000	452.656
31	Kansas City, Mo.	434.829	322.581	384.308
37	Virginia Beach, Va	393.089	270.270	336.015
49	Toledo	332.943	204.082	271.639
61	Arlington Texas	261.721	163.934	230.205
73	Baton Rouge, La.	219.531	136.986	201.033
85	Hialeah, Fla.	188.008	117.647	179.243
97	Bakersfield, Calif.	174.820	103.093	162.270

10.2.4 Benford's Law

Benford's law (known also as the first digit law, first digit phenomenon or leading digit phenomenon) states that the distribution of the first digit is not uniform. This law applies, if the numbers under investigation are not entirely random, but somehow socially or naturally related. Generally the law applies to data that is not dimensionless, meaning that the probability distribution of the data is invariant under a change of scale, and data, that is selected out of a variety of different sources. Benford's law does not apply to uniform distributions, like lottery numbers [11,20,21].

Benford's law results that the probability of a number to be the first digit in tables of statistics , listings, etc., is biggest for one and smallest for nine [11,20].

Many physical laws and human made systems require cutoffs to number series, for example the street addresses begin from 1 and end up to some cutoff value. The cutoffs impose these systems to obey Benford's law, which can be presented for a digit D (1, 2,. . . ,9) by the logarithmic distribution:

$$P_D = \log_{10}(1 + 1/D) \tag{10.9}$$

The base number of logarithm can also be other than 10 [21]. According to this equation the probabilities for digits 1,...,9 lie between 0.301,...,0.045, as shown in Fig. 10.2. The probability for 1 is over 6 times greater than for 9 [20].

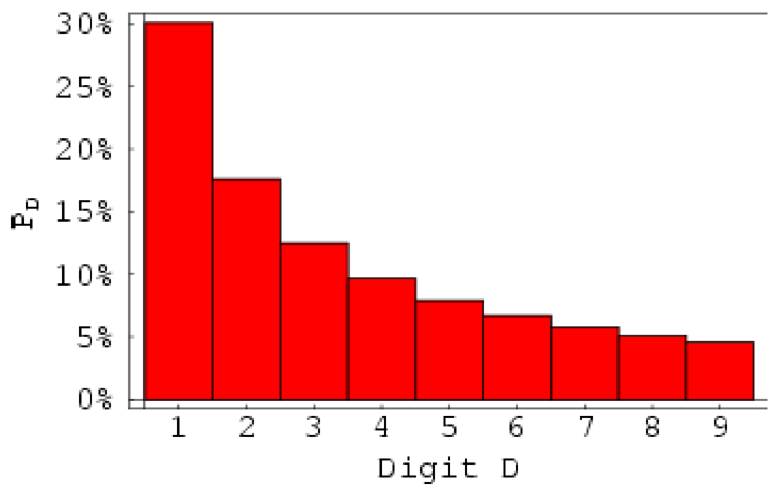

Figure 10.2: Distribution of Benford's law for digits $1, \ldots, 9$ [20]

The examples of Benford's law are numerous, including addresses, the area and basin of rivers, population, constants, pressure, molecular and atomic weight, the half-lives of radioactive atoms, cost data, powers and square root of whole numbers, death rate, budget data of corporations, income tax and even computer bugs [11,20,21].

Benford's law is a powerful and relatively simple tool for pointing suspicion at frauds, embezzlers, tax evaders and sloppy accountants. The income tax agencies of several nations and several states, including California, are using detection software based on Benford's law, as are a score of large companies and accounting businesses [21].

The social and natural impact affects all the listed examples. Budget data is affected by the corporation size, particular industry a company belongs to, the quality of the management, the state of the market, etc. The size of a river basin depends on a depth and breadth of the river. Most of the dependencies can be approximated by simple formulas: Linear, power or exponential, oscillating, leading to saturation.

10.2.5 Fractal dimension

Fractals are characterised by three consepts: Self-similarity, response of measure to scale, and the recursive subdivision of space. Fractal dimension can be measured by many different types of methods. Similar to all these method is, that they all rely heavily on the power law when plotted to logarithmic scale, which is the property relating fractal dimension to power laws [22,23].

One definition of fractal dimension D is the following equation:

$$D = \log_{10} N / \log_{10}(1/R), \tag{10.10}$$

where N is the number of segments created, when dividing an object, and R is the length of each of segments.

This equation relates to power laws as follows:

$$\log(N) = D \cdot \log(1/R) = \log(R^{-D}), \tag{10.11}$$

so that

$$N = R^{-D}. \tag{10.12}$$

It is simple to obtain a formula for the dimension of any object provided. The procedure is just to determine in how many parts it gets divided up into (N) when we reduce its linear size, or scale it down ($1/R$).

By applying the equation to line, square and cubicle, we get the following results; For a line divided in 4 parts, N is 4 and R is $1/4$, so dimension $D = \log(4)/\log(4) = 1$. For a square divided in four parts N is 4, R is $1/2$, and dimension $D = log(4)/log(2) = 2 \cdot \log(2)/\log(2) = 2$. And for a cubicle divided in 8 parts, N is 8, R is $1/2$ and dimension $D = log(8)/log(2) = 3 \cdot log(2)/log(2) = 3$.

The following series of pictures (Fig. 10.3)represents iteration of the *Koch curve*. By applying equation (10.11) to the Koch curve, as in Table 10.2, it is evident, that the dimension is not an integer, but instead between 1 and 2.

The dimension is always 1.26185, regardless of the iteration level. Hence,

$$D = \log N / \log(1/R) = 1.26185, \tag{10.13}$$

Which can also be written as

$$N = (1/R)1.26185. \tag{10.14}$$

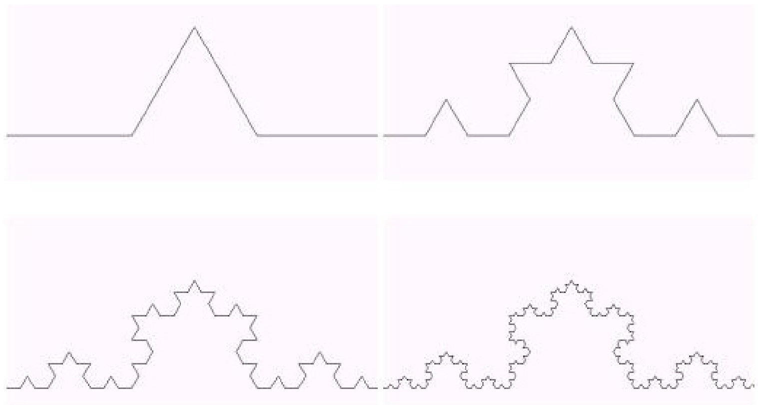

Figure 10.3: The Koch curve

Table 10.2: Statistics of the Koch curve

Iteration number	1	2	3	4	5
Number of segments, N	4	16	64	256	1024
Segment length, R	1/3	1/9	1/27	1/81	1/243
Total length, $N \cdot R$	1.33333	1.77777	2.37037	3.16049	4.21399
$\log N$	0.60206	1.20412	1.80618	2.40824	3.01030
$\log(1/R)$	0.47712	0.95424	1.43136	1.90848	2.38561
Dimension $\log N/\log(1/R)$	1.26185	1.26185	1.26185	1.26185	1.26185

The formulas above indicate that N and R are related through a power law. In general, a power law is a nonlinear relationship, which can be written in the form $N = a(1/R)^D$, where D is normally a non-integer constant and a is a numerical constant which in the case of the Koch curve is 1.26.

Another way of defining the fractal dimension is box counting. In box counting the fractal is put on a grid, which is made of identical squares having size of side h. Then the amount of non-empty squares, k, is counted. The magnification of the method equals to $1/h$ and the fractal dimension is defined by equation: [4]

$$D = \log_{10}(k)/\log_{10}(1/h). \tag{10.15}$$

Also Hausdorff's and Kolmogorov's methods can be used to approximate the fractal dimension. These methods are more accurate, but also harder to use. They are described in [18,24].

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